Harvard University
Math 1b. Second Semester Calculus
Exam One

Name ______________________

March 17, 1999

Exam One

- Do not open this test booklet until you are directed to do so.
- You have 2 hours. The test is out of 100 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please don't put part of the answer to one problem on the back of the sheet for another problem.
- Don't spend too much time on any one problem. Read them all through first and work on them in the order that allows you to make the most progress.
- Be sure to show all of your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Try to be neat. By the same token, be sure to justify your solutions (unless you are explicitly told otherwise), so that the graders can follow your reasoning.
- Good luck!

Problem	Points	Grade
1	10	
2	17	
3	15	
4	15	
5	14	
6	14	
7	15	
Total	100	

Please circle your section:

MWF 10:00	MWF 11:00	MWF 12:00	TTh 10:00	TTh 11:30
Brian	Grisha	Cathy	Andy	Andy
Conrad	Mikhalkin	O'Neil	Engelward	Engelward

1. (10 pts) For each of the following infinite series, determine if it converges or diverges. You need to justify your answer to receive full credit. You don't need to specify whether convergence is conditional or absolute, just whether the series converges or diverges.

a. (2 pts) $\sum_{k=1}^{\infty} e^{\frac{1}{k}}$

b. (2 pts) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$

c. (3 pts) $\sum_{k=1}^{\infty} \frac{|\sin(k)|}{k^4}$

d. (3 pts) $\sum_{k=0}^{\infty} \frac{1}{\sqrt{k^2+1}}$

2. (17 pts)

a. (3 pts each) Evaluate each of the following infinite series.

i. $\sum_{k=0}^{\infty} (-1)^k \, 6^{-2k-2}$

ii. $\sum_{k=0}^{\infty} \frac{2^k \, 5^{k+1}}{3^{k+2} \, 7^k}$

iii. $\sum_{k=3}^{\infty} \frac{1 - 2^k}{3^k}$

b. (4 pts each) For which values of the number a do the following series converge (note the answers are different for each series).

i. $\sum_{k=1}^{\infty} \left(\frac{3}{a}\right)^k$

ii. $\sum_{k=1}^{\infty} \frac{a^k}{5^{2k}}$

3. (15 pts) Let $f(x) = \sum_{k=1}^{\infty} \frac{(x-5)^k}{k \cdot 3^k}$.

a. (5 pts) Find the interval of convergence and radius of convergence for the above Taylor series (be sure to check convergence at endpoints if there are any).

b. (7 pts) Compute the Taylor series expansion about $x = 5$ for $g(x) = \int f(x)dx$, assuming that $g(5) = 0$. Write your answer using $\sum$ notation. In addition, explicitly write out the terms of degree ≤ 4 in this Taylor series (you don't need to simplify the coefficients when you write them down).

c. (3 pts) Find the interval of convergence for the series you computed in part **b**.

4. (15 pts) Let $g(x) = \dfrac{e^x - 1}{x}$ for $x \neq 0$, and suppose that $g(0) = 1$.

a. (4 pts) Using the Taylor expansion for e^x about $x = 0$, show that

$$g(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!}.$$

b. (4 pts) Use the series from part a to show that

$$g'(x) = \sum_{k=1}^{\infty} \frac{k}{(k+1)!} x^{k-1},$$

and check that this expansion for g' has infinite radius of convergence.

c. (5 pts) Explicitly compute $g'(x)$ using the definition of $g(x)$, and use this result to show that $g'(1) = 1$.

d. (2 pts) Explain how the results in parts **a**, **b** and **c** imply that

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} = 1.$$

5. (14 pts) A power series $\sum_{k=0}^{\infty} a_k x^k$ centered at 0 converges for $x = -5$ and diverges for $x = 7$.

a. (2 pts each) Based on this information, determine what happens when x takes on the following values (you should answer either "converges," "diverges," or "cannot be determined from the information given" in each case). You don't need to provide any justification for your answers.

i. $x = 3$

ii. $x = 6$

iii. $x = -10$

iv. $x = 0$

v. $x = -7$

b. (4 pts) Based on the information given, what is the smallest possible value for the radius of convergence for this power series? Be sure to justify your answer.

6. (14 pts) Antoinette wants to compute $\sqrt[3]{9}$, but she doesn't have a calculator with her. She decides to use a degree 2 Taylor polynomial for $f(x) = \sqrt[3]{x}$ centered at $x = 8$.

a. (6 pts) Compute the Taylor polynomial of degree 2 for $\sqrt[3]{x}$ centered at $x = 8$. Write your answer in the form $a_0 + a_1(x-8) + a_2(x-8)^2$, where a_0, a_1, a_2 are numbers which you should write as reduced form fractions (e.g. write $\frac{5}{2 \cdot 3^2}$ or $\frac{5}{18}$ rather than $\frac{200}{6!}$).

b. (3 pts) Use your answer in part **a** to give an approximation to $\sqrt[3]{9}$. You do not need to simplify your answer in this case.

c. (5 pts) Show that the absolute value of the difference between the true value of $\sqrt[3]{9}$ and your answer in part **b** is $\leq .005 = \frac{1}{2} \cdot 10^{-2}$.

7. (15 pts) Determine the radius and interval of convergence for each of the following power series. Be sure to check convergence at the endpoints of the interval (if there are any endpoints).

a. (8 pts) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^3} x^k$

b. (7 pts) $\displaystyle\sum_{k=1}^{\infty} \frac{x^{3k}}{(k+3)!}$