Harvard University
Math 1b
Spring, 2000

Your Name: ____________________
Your TF's Name: ________________
April 17, 2000

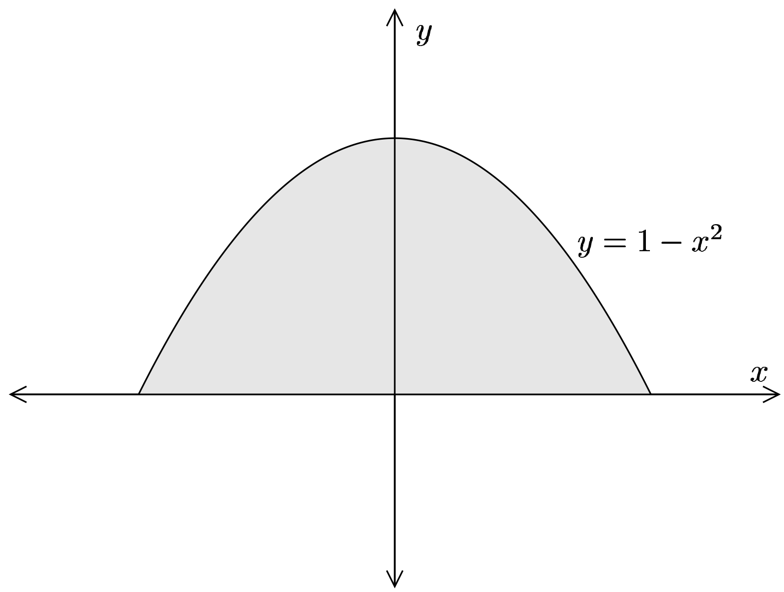
# Second Exam

- Do not open this exam booklet until you are directed to do so.
- You have 120 minutes to earn 100 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem (and explain clearly that you are doing so). Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read through them all first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given (unless indicated otherwise). You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- No calculators are allowed on this exam.
- Good luck!

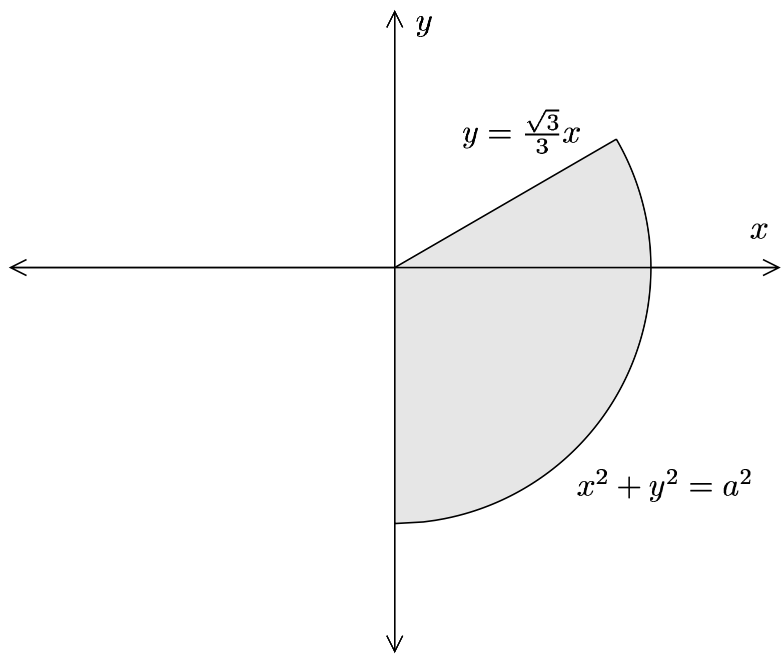
| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 15 | |
| Total | 100 | |

**1.** (15 points) An aqueduct that runs flat along the ground has vertical cross sections in the shape of the upside-down parabola $y = 1 - x^2$ (as depicted below). At the end of the aqueduct, there is a metal door which is also in the shape of $y = 1 - x^2$, with its base along the the $x$-axis. If the aqueduct is completely full of a liquid with weight density $\rho = 100 \frac{\text{N}}{\text{m}^3}$, what is the total fluid force acting on the door? (All measurements are in meters, so that the top of the aqueduct, for example, stands 1 meter high.)

Remember to show your work in setting up the integral. You need a correct integral as well as the correct numerical answer to receive full credit.



**2.** (15 points) The region depicted below is bounded by the circle $x^2 + y^2 = a^2$, by the $y$-axis, and by the line $y = \frac{\sqrt{3}}{3}x$. Find the volume of the object created by revolving this region about the $y$-axis. (This object will look something like a sphere with a cone removed from it.) Be sure to show all your work, including indicating how you are making your slices to find the volume and writing down the integral that you have to evaluate to obtain your final answer. Finally, be sure to evaluate the integral and find a numerical answer (which may be left in terms of $a$ and $\pi$).



**3.** (15 points) Evaluate the following integrals. (Don't forget to include constants of integration for the indefinite integrals.) (Each part is worth 5 points.)

3a. $\displaystyle\int (\ln x)^2\, dx$

3b. $\displaystyle\int_0^1 \frac{e^{3\tan^{-1} x}}{1+x^2}\, dx$

3c. $\displaystyle\int x^2 \cos x\, dx$

**4.** (15 points) Evaluate the following integrals. (Don't forget to include constants of integration for the indefinite integrals.) (Each part is worth 5 points.)

4a. $\displaystyle\int_0^1 \frac{x(x+3)}{(1+x^2)(x+1)}\,dx$

4b. $\displaystyle\int \frac{2}{x^3-x}\,dx$

4c. $\displaystyle\int \frac{x^4+3x^3+3x^2+2x-1}{x^3+2x^2+x}\,dx$

**5.** (10 points) For each of the following improper integrals, compute it if it converges, and otherwise explain why it diverges.

5a. (3 points)

$$\int_0^1 \frac{1}{x^{1/3}}\,dx$$

5b. (3 points)

$$\int_{-\infty}^0 e^{-x^2} x\,dx$$

5c. (4 points)

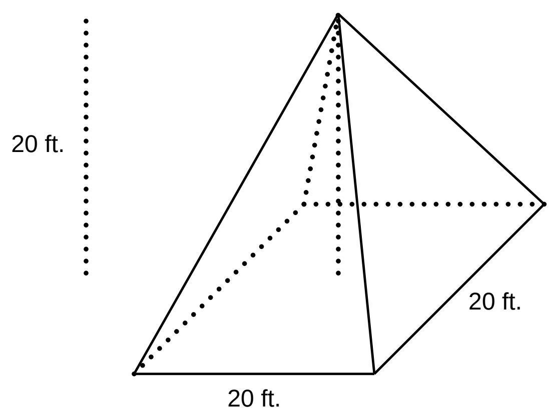
$$\int_0^\pi \tan x\,dx$$

**6.** (15 points) Because of his fondness for ancient Egypt, Farmer Bob stores his grain inside a pyramid with a square base. Each side of the square base measures 20 feet in length, and the height of the pyramid is also 20 feet. (See the figure.)

One evening, a flying saucer full of aliens visits Farmer Bob's farm. After spending a few hours forming crop circles in the corn, the aliens decide to investigate Farmer Bob's grain. To do so, they cut a tiny hole in the top of the pyramid, and suck all of the grain out through that hole.

What is the total work required to lift all of the grain out of the pyramid through the small hole at the top, expressed in foot-pounds? (Assume that the pyramid is initially completely full of grain, and that grain weighs 25 pounds per cubic foot.)

Be sure to indicate how you are slicing the pyramid, and show all other work that you do. To receive full credit for this problem, you must obtain an integral that represents the amount of work needed to be done (with justification) and you must evaluate the integral to find a numerical value for the amount of work done.



**7.** (15 points)
7a. (5 points) Find an integral representing the arclength of the curve given parametrically by $x = a \cos t$ and $y = b \sin t$ for $0 \leq t \leq 2\pi$. (You should not evaluate the integral that you get; just write it down.)

7b. (5 points) Find an integral representing the surface area of the solid created by revolving the segment of the curve $y = x + 1$ from $x = 0$ to $x = 2$ about the $x$ axis. Evaluate the integral you obtain.

7c. (5 points) Find an integral representing the volume of the solid created by revolving the portion of the curve $y = \frac{1}{x^{2/3}}$ with $2 \leq x \leq \infty$ about the $x$-axis. Tell whether the resulting integral is convergent or divergent. If it is convergent, evaluate it, and if it is divergent, explain why it diverges.