Exam Two Answers (Spring 2000)

1. You slice perpindicular to the y axis, and your slices run from $y = 0$ up to $y = 1$. For any given y value, the depth you are at is $h = 1 - y$, the weight density is $\rho = 100$, and the area of a slice is given by $2\sqrt{1-y}\Delta y$. This area comes from solving the equation $y = 1 - x^2$ for x to get $y = \pm\sqrt{1-y}$. The width of a slice is then $w(y) = \sqrt{1-y} - \left(-\sqrt{1-y}\right)$, or $w(y) = 2\sqrt{1-y}$. Multiplying this by the thickness, Δy, then gives the surface area of the slice.

Summing up all these little slices and passing to a limit to form an integral, we get

$$\begin{aligned}\text{Surface Pressure} &= \int_0^1 \rho h(y) w(y)\, dy \\ &= \int_0^1 100(1-y)\left(2\sqrt{1-y}\right)\, dy \\ &= 200\int_0^1 (1-y)^{\frac{3}{2}}\, dy \\ &= -200 \cdot \frac{2}{5}(1-y)^{\frac{5}{2}}\Big|_0^1 \\ &= 80.\end{aligned}$$

2. First of all, split the region into two parts: one below the x-axis, and one above. The one above will have cross-sections perpindicular to the y-axis which look like annuli, while in the one below they will just be solid discs. For both regions we will, of course, slice perpindicular to the y-axis.

For the region above, we have to figure out where the line $y = \frac{\sqrt{3}}{3}x$ intersects $x^2 + y^2 = a^2$. The equation for the line can be rewritten $x = \sqrt{3}y$, and plugging this into the equation for the circle gives $\left(\sqrt{3}y\right)^2 + y^2 = a^2$, which is the same as $4y^2 = a^2$, and so $y = a/2$ at the point of intersection of the line and the circle. Thus, for the above region, we will be slicing from $y = 0$ up to $y = a/2$. In this region, the cross-sections are annuli which ahve an inner radius of $x = \sqrt{3}y$ (from the equation for the line) and an outer radius of $x = \sqrt{a^2 - y^2}$ (from the equation for the circle). Thus the volume is given by

$$\begin{aligned}\int_0^{a/2}\left(\pi\left(\sqrt{a^2-y^2}\right)^2 - \pi\left(\sqrt{3}y\right)^2\right) dy &= \pi\int_0^{a/2}(a^2 - 4y^2)\, dy \\ &= \pi\frac{a^3}{2} - \pi\frac{a^3}{6} = \pi\frac{a^3}{3}.\end{aligned}$$

For the region below, the integral runs from $y = -a$ up to $y = 0$, and the radius of each disc is given by $\sqrt{a^2 - y^2}$. Thus the volume is given by

$$\int_{-a}^{0} \pi \left(\sqrt{a^2 - y^2}\right)^2 dy = \pi \left(a^2 y - \frac{y^3}{3}\right)\Big|_{-a}^{0}$$
$$= \pi \frac{2a^3}{3}.$$

Adding together the results from the upper and lower regions, we see that the answer is πa^3.

3.

3a. Start using integration by parts with $u = (\ln x)^2$ and $dv = dx$, so that $v = x$ and $du = 2(\ln x)\frac{1}{x}dx$. Thus,

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2\int \ln x\, dx.$$

Now use integration by parts again to evaluate $\int \ln x\, dx$, so that $u = \ln x$ and $dv = dx$, so $du = \frac{1}{x}dx$ and $v = x$. Thus,

$$\int \ln x\, dx = x \ln x - \int dx = x \ln x - x + C.$$

Combining these results, we see that

$$\int (\ln x)^2\, dx = x(\ln x)^2 - 2x \ln x + 2x + C.$$

3b. Use the substitution $u = 3\tan^{-1} x$ to get

$$\int \frac{e^{3\tan^{-1} x}}{1 + x^2}\, dx = \frac{1}{3}\int e^u\, du,$$

since $du = 3\frac{1}{1+x^2}dx$. Thus we have

$$\int_0^1 \frac{e^{-\arctan x}}{1 + x^2}\, dx = \frac{1}{3} e^{3\tan^{-1} x}\Big|_0^1 = \frac{1}{3}(e^{3\pi/4} - e^0) = \frac{e^{3\pi/4} - 1}{3}.$$

3c. Using integration by parts twice, we see that

$$\int x^2 \cos x\, dx = x^2 \sin x - \int (2x) \sin x\, dx$$
$$= x^2 \sin x - 2\left(-x\cos x + \int \cos x\, dx\right)$$
$$= x^2 \sin x + 2x\cos x - 2\sin x + C$$

4.

4a. The partial fraction decomposition is

$$\frac{x(x+3)}{(1+x^2)(x+1)} = \frac{1+2x}{1+x^2} + \frac{-1}{x+1},$$

and so the indefinite integral is seen to be

$$\begin{aligned}\int \frac{1+2x}{1+x^2}\,dx - \int \frac{1}{x+1}\,dx &= \int \frac{2x}{1+x^2}\,dx + \int \frac{1}{1+x^2}\,dx - \int \frac{1}{x+1}\,dx \\ &= \ln|1+x^2| + \tan^{-1} x - \ln|x+1| + C.\end{aligned}$$

The definite integral is now easily evaluated by plugging in the limits, and so

$$\int_0^1 \frac{x(x+3)}{(1+x^2)(x+1)}\,dx = \frac{\pi}{4}.$$

4b. The denominator factors as $x(x-1)(x+1)$, and so the partial fraction decomposition is seen to be

$$\frac{2}{x^3-x} = \frac{-2}{x} + \frac{1}{x-1} + \frac{1}{x+1}.$$

Thus the integral is simply

$$\int \frac{2}{x^3-x}\,dx = -2\ln|x| + \ln|x-1| + \ln|x+1| + C.$$

4c. For this, you have to use long division and then convert using partial fractions. The long division yields

$$\frac{x^4+3x^3+3x^2+2x-1}{x^3+2x^2+x} = x+1+\frac{x-1}{x^3+2x^2+x}.$$

Partial fraction analysis then shows that

$$\frac{x-1}{x^3+2x^2+x} = \frac{-1}{x} + \frac{1}{x+1} + \frac{2}{(x+1)^2}.$$

Thus we see that the indefinite integral is

$$\frac{1}{2}x^2 + x - \ln|x| + \ln|x+1| - \frac{2}{x+1}.$$

5.

5a. We have

$$\int_0^1 \frac{1}{x^{\frac{1}{3}}}\,dx = \lim_{h\to 0+} \frac{3}{2}x^{\frac{2}{3}}\Big|_{x=h}^{x=1} = \frac{3}{2}.$$

5b. Using the substitution $u = x^2$, we see that

$$\int e^{-x^2} x\,dx = -\frac{1}{2}e^{-x^2} + C.$$

Thus we see that

$$\int_{-\infty}^{0} e^{-x^2} x\,dx = \lim_{h\to -\infty} -\frac{1}{2}e^{-x^2}\Big|_h^0 = -\frac{1}{2}.$$

5c. We have

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C,$$

where we have used the substitution $u = \cos x$ in order to obtain the last step. Also, we see that the integrand becomes infinite at $x = \pi/2$. Thus, since

$$\int_0^{\pi/2} \tan x\,dx = \lim_{h\to \frac{\pi}{2}+} -\ln|\cos x|\big|_0^h = \infty,$$

we see that the integral must diverge.

6. Let x be a variable which measures the distance down from the top of the pyramid. Slice perpindicular to this x axis. At any height x, the area of a slice can be found by similar triangles to be x^2. Indeed, such a slice looks like a square, and the side length of the square varies from 0 at the top to 20 at the bottom (moving linearly from where $x = 0$ to where $x = 20$). Thus proportions or similar triangles can show that the side length at a level x is also x and so the area of a square at the level x is x^2. The distance which you have to move a slice made a level x is also exactly x. Thus, the total work done is given by

$$\int_0^{20} (\text{weight density}) \cdot \text{distance} \cdot \text{distance}\,dx = \int_0^{20} 25x^3\,dx = 1,000,000.$$

7.

7a. Since $x'(t) = -a \sin t$ and $y'(t) = b \cos t$, our integral is

$$\int_0^{2\pi} \sqrt{a^2 \sin^2 t + b^2 \cos^2 t}\, dt.$$

7b. The integral is

$$\begin{aligned} \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx &= \int_0^2 2\pi(x+1)\sqrt{1+1}\, dx \\ &= \sqrt{2}\pi(x+1)^2 \Big|_0^2 \\ &= 8\pi\sqrt{2}. \end{aligned}$$

7c. The integral is

$$\int_2^\infty \pi \left(\frac{1}{x^{2/3}}\right)^2 dx = \int_2^\infty \pi x^{-4/3}\, dx.$$

This integral is equal to

$$\lim_{h\to\infty} -3\pi x^{-1/3}\Big|_2^h = \frac{3\pi}{\sqrt[3]{2}} = \frac{3\pi\sqrt[3]{4}}{2}.$$