Problem Set #20

2. (a) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

20. The series $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n}$ has positive terms and $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)^2}{2^{n+1}}\cdot\dfrac{2^n}{n^2}\right] = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2\cdot\dfrac{1}{2} = \dfrac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

22. Using the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-3)^{n+1}/(n+1)!}{(-3)^n/n!}\right| = 3\lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0 < 1$, so the series is absolutely convergent.

31. (a) $\lim\limits_{n\to\infty}\left|\dfrac{1/(n+1)^3}{1/n^3}\right| = \lim\limits_{n\to\infty}\dfrac{n^3}{(n+1)^3} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim\limits_{n\to\infty}\left|\dfrac{(n+1)}{2^{n+1}}\cdot\dfrac{2^n}{n}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{2n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{2}+\dfrac{1}{2n}\right) = \dfrac{1}{2}$. Conclusive (convergent).

(c) $\lim\limits_{n\to\infty}\left|\dfrac{(-3)^n}{\sqrt{n+1}}\cdot\dfrac{\sqrt{n}}{(-3)^{n-1}}\right| = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{n}{n+1}} = 3\lim\limits_{n\to\infty}\sqrt{\dfrac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim\limits_{n\to\infty}\left|\dfrac{\sqrt{n+1}}{1+(n+1)^2}\cdot\dfrac{1+n^2}{\sqrt{n}}\right| = \lim\limits_{n\to\infty}\left[\sqrt{1+\dfrac{1}{n}}\cdot\dfrac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive.

33. (a) $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim\limits_{n\to\infty}\dfrac{x^n}{n!} = 0$ by Theorem 8.2.6.

17. $a_n = (-1)^n \dfrac{x^{2n}}{(2n)!}$ $\qquad a_{n+1} = (-1)^{n+1} \dfrac{x^{2(n+1)}}{(2(n+1))!} = (\ )^{n+1} \dfrac{x^{2n+2}}{(2n+2)!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \left| \frac{x^2 \cdot (2n)!}{(2n+2)!} \right|$$

Note that $(2n+2)! = (2n)!(2n+1)(2n+2)$, by definition, so we have:

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+1)(2n+2)} \right|$$

$$= 0$$

So $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ converges for all x, by the ratio test.

18. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n}$ is geometric, with $r = \dfrac{(x-3)}{5}$, so the series should converge when $|r| < 1$

$$\Rightarrow \quad -1 < \frac{x-3}{5} < 1$$

$$\Rightarrow \quad -5 < x-3 < 5 \quad \Rightarrow \quad -2 < x < 8$$

Now, applying the ratio test, we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-3)^{n+1}}{5^{n+1}}}{\frac{(x-3)^n}{5^n}} \right| = \lim_{n\to\infty} \left| \frac{(x-3)}{5} \right| = \left| \frac{x-3}{5} \right| < 1$$

So $-1 < \dfrac{x-3}{5} < 1$, exactly ~~what~~ we got by applying what we know about geometric series!