Problem Set 3

16. (a) $\dfrac{x-1}{x^3+x^2} = \dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

(b) $\dfrac{x-1}{x^3+x} = \dfrac{x-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

18. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2)+B(x+1)$. Substituting -2 for x gives $-3=-B \Leftrightarrow B=3$. Substituting -1 for x gives $-2=A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right) dx = \Big[-2\ln|x+1| + 3\ln|x+2|\Big]_0^1$$
$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

24. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2-x+6 = A(x^2+3)+(Bx+C)x$. Substituting 0 for x gives $6=3A \Leftrightarrow A=2$. The coefficients of the x^2-terms must be equal, so $1=A+B \Rightarrow B=1-2=-1$. The coefficients of the x-terms must be equal, so $-1=C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right) dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right) dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

27.

$$\begin{array}{r|l} & x \\ \hline x^2+1 & x^3 \\ & x^3+x \\ \hline & -x \end{array}$$

By long division, $\dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$. Thus,

$$\int_0^1 \frac{x^3}{x^2+1}\,dx = \int_0^1 x\,dx - \int_0^1 \frac{x\,dx}{x^2+1}$$
$$= \left[\tfrac{1}{2}x^2\right]_0^1 - \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du \quad [\text{where } u = x^2+1,\ du = 2x\,dx]$$
$$= \tfrac{1}{2} - \left[\tfrac{1}{2}\ln u\right]_1^2 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2 = \tfrac{1}{2}(1-\ln 2)$$

29. Let $u=\sqrt{x}$, so $u^2=x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1+\frac{4}{u^2-4}\right) du \quad [\text{by long division}]$$
$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)}.$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so

$\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and the last integral is

$$2+8\int_3^4 \left(\frac{-1/4}{u+2} + \frac{1/4}{u-2}\right) du = 2 + 8\left[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\right]_3^4$$
$$= 2 + [2\ln|u-2| - 2\ln|u+2|]_3^4 = 2 + 2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4$$
$$= 2 + 2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2 + 2\ln\tfrac{2/6}{1/5}$$

3. $\int \sin^2\theta\, d\theta = \frac{1}{2}\int(1-\cos 2\theta)\, d\theta = \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta\, d\theta$

Integration Handout

$$= \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + C$$

$$= \boxed{\frac{\theta}{2} - \frac{\sin 2\theta}{4} + C}$$

4. a) $x = 3\sin t \Rightarrow dx = 3\cos t\, dt$

b) $x = 3\sin t \Rightarrow \sqrt{9-x^2} = \sqrt{9 - 9\sin^2 t} = \sqrt{9(1-\sin^2 t)} = \sqrt{9\cos^2 t} = 3\cos t$

c) $0 = 3\sin t \Rightarrow t = 0$, $3 = 3\sin t \Rightarrow \sin t = 1 \Rightarrow t = \pi/2$

d) $\int_0^3 \sqrt{9-x^2}\, dx = \int_0^{\pi/2} \left(\sqrt{9 - 9\sin^2 t} \cdot 3\cos t\right) dt$

e) $= \int_0^{\pi/2} 3\cos t \cdot 3\cos t\, dt = 9\int_0^{\pi/2} \cos^2 t\, dt = 9 \cdot \frac{1}{2}\int_0^{\pi/2} (1+\cos 2t)\, dt$

$$= \frac{9}{2}\left[t \Big|_0^{\pi/2}\right] + \frac{9}{4}\left[\sin 2t \Big|_0^{\pi/2}\right] = \frac{9\pi}{4} + \frac{9}{4}(\sin\pi - \sin 0) = \frac{9\pi}{4}$$

f) $\int_0^3 \sqrt{9-x^2}\, dx$ is the area of a quarter circle of radius 3.

Therefore, the area of the entire circle is $A = 4\int_0^3 \sqrt{9-x^2}\, dx$

$$= 4\left(\frac{9\pi}{4}\right) = 9\pi .$$