Series: Convergence and Divergence

Here is a compilation of what we have done so far (before discussing Taylor polynomials) in terms of convergence and divergence.

- Series that we know about:

  Geometric Series: A geometric series is a series of the form $\sum_{n=0}^{\infty} ar^n$. The series converges if $|r| < 1$ and diverges otherwise[1]. If $|r| < 1$, the sum of the entire series is $\frac{a_1}{1-r}$ where a is the first term of the series and r is the common ratio.

  p-Series Test: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise[2].

- *Nth Term Test for Divergence:* If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

  Note: If $\lim_{n\to\infty} a_n = 0$ we know nothing. It is possible that the series converges but it is possible that the series diverges.

Comparison Tests:

These tests can be applied only to series in which all the terms are positive. (They are based on the idea of bounded the monotonic sequence of partial sums.)

- *Direct Comparison Test:* If a series $\sum_{n=1}^{\infty} a_n$ has all positive terms, and all of its terms are eventually bigger than those in a series that is known to be divergent, then it is also divergent. The reverse is also true–if all the terms are eventually smaller than those of some convergent series, then the series is convergent.

  That is, suppose $\sum a_n$, $\sum b_n$ and $\sum c_n$ are all series with positive terms and $a_n \leq b_n \leq c_n$ for all n sufficiently large, then

  if $\sum c_n$ converges, then $\sum b_n$ does as well
  if $\sum a_n$ diverges, then $\sum b_n$ does as well.

  (This is a good test to use with rational functions. Specifically, if the degree of the denominator is more than 1 greater than the degree of the numerator, try to prove that the series converges (compare with a p-series). In other cases, including when the difference in degree is exactly 1, prove that it diverges).

- *Limit Comparison Test:* Use this when you know what you want to compare to but the inequalities go the wrong way. Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is finite and non-zero, then either both the series converge or both the series diverge.

- *Integral Test:* Let $a_n = f(n)$. Then, if f is continuous, decreasing, and positive on $[1, \infty)$, we have that $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

  This test was instrumental in proving the p-series result.

- *General Point:* Multiplying by constants and leaving off the first few terms have *no effect* on whether a series converges. Also, adding a convergent series to another series will not change whether the other one converges.

How to approach a series:

1. Is the series a geometric series or a $p-$series? If so, you can draw a conclusion.
2. If the series is neither a geometric series nor a $p-$series but looks like one of these and all the terms are positive (or eventually positive), try direct comparison or limit comparison.
3. See if the terms of the series tend towards zero. If they don't then the series diverges.
4. If all else fails and the appropriate integral looks tractable try the Integral Test.

Keep track of what you're doing by calling upon the test you are using. Suppose you compute a limit and the limit is 3. What conclusion you draw depends upon what this is the limit of. If the partial sums are tending towards 3 then the series converges to 3. If the terms are tending towards 3 then the series diverges. If the limit of the terms of one series to the terms of another is 3 then the series either both converge or both diverge.

[1] We proved this by writing the partial sums in closed form and computing a limit.
[2] We proved this using the Integral Test