Name: ______________________________ ID#: ____________________

Solutions to Final Exam

Math 1b
Calculus, Series, Differential Equations

27 January 2004

Show all of your work. Full credit may not be given for an answer alone. You may use the backs of the pages or the extra pages for scratch work. Do not unstaple or remove pages.

This is a non-calculator exam.

Please circle your section:

MWF9	Matthew Leingang	TΘ10	Andrew Lobb
MWF10	Ken Chung	TΘ10	Chun-Chun Wu
MWF10	Florian Herzig	TΘ11:30	Amanda Alvine
MWF10	Michael Schein	TΘ11:30	Rosa Sena-Dias
MWF11	Matthew Leingang		
MWF11	Janet Chen		
MWF12	Ken Chung		

Students who, for whatever reason, submit work not their own will ordinarily be required to withdraw from the College.

—*Handbook for Students*

1. (15 Points) *Test the following for convergence or divergence. Give justification.*

(i) $\sum_{n=1}^{\infty} \frac{1}{n^{17}}$

Solution. This is a p-series with $p = 17 > 1$. So the series converges. □

(ii) $\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

Solution. The ratio of successive terms is

$$\frac{\frac{(n+1)^2}{e^{n+1}}}{\frac{n^2}{e^n}} = \left(\frac{n+1}{n}\right)^2 \frac{1}{e} \longrightarrow \frac{1}{e},$$

so the series converges by the ratio test. □

(iii) $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

Solution. To employ the alternating series test, we must check three things: whether $\left\{\sin\left(\frac{1}{n}\right)\right\}$ is positive, decreasing, and tends to zero. The first is clear because $\sin x$ is positive when x is in between 0 and $\frac{\pi}{2}$, and this includes the set of all $\frac{1}{n}$ where n is a positive integer. The second can be checked by differentiation:

$$\frac{d}{dx} \sin\left(\frac{1}{x}\right) = -\cos\left(\frac{1}{x}\right)\frac{1}{x^2},$$

which is less than zero if $0 \le x \le \frac{\pi}{2}$. To check the limit, we note that

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty} \frac{1}{n}\right) = \sin 0 = 0.$$

So the series converges by the alternating series test. Incidentally, the series does not converge absolutely. The limit comparison test with $\left\{\frac{1}{n}\right\}$ confirms that. □

______ / 15

2. (9 Points) *Every 5 minutes Mattia (age 10 months) eats 1/3 of the Cheerios on her tray and throws the rest on the floor.*[1] *Her father gathers the remaining Cheerios off the floor and puts them back on her tray, where she continues to eat a third and toss the rest.*

Assume she is given 162 Cheerios at 8:30am. Fill in the following table:[2]

Solution. This problem is much easier than it seems. If $\{c_n\}$ denotes the number of Cheerios on the floor at time step n, measured in 5-minute intervals from 8:30am, then

$$c_{n+1} = \frac{2}{3}c_n.$$

Therefore we can fill out the table as follows:

Time	Cheerios on the floor
8:35	$162\left(\frac{2}{3}\right) = 108$
8:40	$108\left(\frac{2}{3}\right) = 72$
8:45	$72\left(\frac{2}{3}\right) = 48$
...	...
9:30	$162\left(\frac{2}{3}\right)^{12} = \frac{2^{13}}{3^8}$

□

[1]Based on a true story.
[2]You may assume that Cheerios are infinitely divisible. This has been experientially verified.

______ / 9

3. (7 Points)

(a) (2 points) Write down the formula for the sum of a geometric series:

$$a + ar + ar^2 + ar^3 + \cdots = ?$$

For which a and r is the formula valid?

Solution. The series converges to

$$\frac{a}{1-r}$$

provided $|r| < 1$. □

(b) (5 points) Find a power series centered at zero for the function $g(x) = \ln(1+3x)$. What is its radius of convergence?

Solution. Notice that

$$\frac{1}{1+3x} = \frac{1}{1-(-3x)} = \sum_{n=0}^{\infty} (-1)^n 3^n x^n.$$

This series converges as long as $|-3x| < 1$, which is true if and only if $|x| < \frac{1}{3}$. So the radius of convergence of this series is $\frac{1}{3}$. Now

$$\begin{aligned} \ln(1+3x) &= \int \frac{3}{1+3x}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n 3^{n+1} x^n \right) dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{n+1} x^{n+1} + C. \end{aligned}$$

Because $g(0) = \ln 1 = 0$, we must have that $C = 0$, and so

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{n+1}}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 3^n}{n} x^n.$$

□

_____ / 7

4. (8 Points) *Find a number* N *such that* $S_N = \sum_{n=1}^{N} (-1)^n \frac{1}{n^2}$ *is within* 10^{-10} *of* $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$. *Make sure you verify the conditions of any theorem you use.*

Solution. We will employ the remainder test for alternating series, and to do so we must first check the conditions of the alternating series test, namely that $\left\{\frac{1}{n}\right\}$ is positive and decreases down to zero. This is clear.

The remainder estimate says that

$$|R_N| \leq \frac{1}{(N+1)^2},$$

So to make sure that $|R_N| \leq 10^{-10}$, we need only choose N such that

$$\frac{1}{(N+1)^2} \leq \frac{1}{10^{10}}.$$

This is true as long as $N \geq 10^5 - 1 = 9999$. □

_____ / 8

5. (10 Points) *Find* $\int e^{2x} \sin 3x \, dx$.

Solution. First let $u = \sin 3x$ and $dv = e^{2x}\, dx$ and integrate by parts:

$$\int e^{2x} \sin 3x \, dx = \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx.$$

Now let $u = \cos 3x$ and $dv = e^{2x}, dx$ and integrate by parts again.

$$\begin{aligned} \int e^{2x} \sin 3x \, dx &= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \left\{ \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx \right\} \\ &= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x - \frac{9}{4} \int e^{2x} \sin 3x \, dx. \end{aligned}$$

Thus

$$\begin{aligned} \frac{13}{4} \int e^{2x} \sin 3x \, dx &= \frac{1}{2} e^{2x} \sin 3x - \frac{3}{4} e^{2x} \cos 3x \\ \int e^{2x} \sin 3x \, dx &= \frac{2}{13} e^{2x} \sin 3x - \frac{3}{13} e^{2x} \cos 3x. \end{aligned}$$

□

______ / 10

6. (9 Points) *In each of the following, determine a substitution which makes the integral easier. Transform the integral to make it completely in terms of the new variable. You need not carry out the integration any further than this.*

(i) $\int_0^1 \frac{e^x\,dx}{e^{2x}+e^{3x}}$

Solution. Let $u = e^x$. Then $du = e^x\,dx$. The limits of integration change to $u(0) = 1$ and $u(1) = e$. So we have

$$\int_0^1 \frac{e^x\,dx}{e^{2x}+e^{3x}} = \int_1^e \frac{du}{u^2+u^3}.$$

□

(ii) $\int (u^2+1)\sqrt{u^2+9}\,du$

Solution. Let $u = 3\tan\theta$. Then $du = 3\sec^2\theta\,d\theta$, and $\sqrt{u^2+9} = 3\sec\theta$. The integral becomes

$$\int (u^2+1)\sqrt{u^2+9}\,du = \int (9\tan^2\theta+1)\sec^3\theta\,d\theta.$$

□

(iii) $\int \frac{(\ln t)^{46}}{t}\,dt$

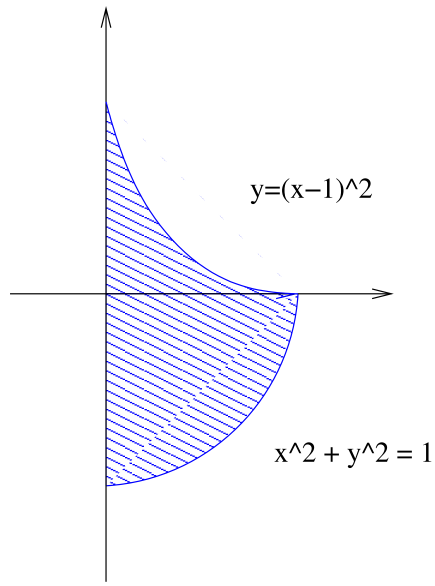
Solution. Let $u = \ln t$, so $du = \frac{1}{t}\,dt$. We have

$$\int \frac{(\ln t)^{46}}{t}\,dt = \int u^{46}\,du.$$

□

_____ / 9

7. (9 Points) *Let R be the region bounded by the y-axis, the curve $y = (x-1)^2$, and the curve $x^2 + y^2 = 1$.*



What is the volume of the "spinning top" generated by rotating R about the y-axis? (You can check the volume of the bottom half of the solid against a well-known formula. However, your answer should use an integral to prove this answer.)

Solution. We use the method of cylindrical shells to get

$$V = \int_0^1 2\pi x \left((x-1)^2 + \sqrt{1-x^2} \right) \, dx.$$

Now

$$\begin{aligned} 2\pi \int_0^1 x(x-1)^2 \, dx &= 2\pi \int_0^1 \left(x^3 - 2x^2 + x \right) \, dx \\ &= 2\pi \left[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 \right]_0^1 \\ &= 2\pi \left(\frac{1}{4} - \frac{2}{3} + \frac{1}{2} \right) = \frac{\pi}{6}. \end{aligned}$$

For the second, let $u = 1 - x^2$. Then $du = -2x \, dx$ and

$$\begin{aligned} 2\pi \int_0^1 x\sqrt{1-x^2} \, dx &= -\pi \int_1^0 \sqrt{u} \, du \\ &= \left[\frac{2\pi}{3} u^{3/2} \right]_0^1 = \frac{2\pi}{3}. \end{aligned}$$

(This jives with the formula that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$).
Hence the volume of the whole spinning top is $\frac{\pi}{6} + \frac{2\pi}{3} = \frac{3\pi}{4}$. □

_____ / 9

8. (19 Points) *The city Erauqs has a very homogeneous population. Within its square city limits (4 miles on each side) the population is spread evenly, with constant population density* $\rho = 10,500$ *people per square mile So the entire city has a population of*

$$(\textit{Area})(\textit{pop. dens.}) = (16 \textit{ sq. mi.})(10,500 \textit{ people/sq. mi.}) = 168,000 \textit{ people}$$

The nearby metropolis Elcric is more densely populated near the city center. Elcric occupies a circular piece of land with radius 10 miles, and the population density is given by the function $\rho(r) = Ce^{-r^2}$ *people per square mile, where* r *is the distance from the center of Elcric in miles, and* $C = \frac{8.3\times 10^7}{\pi(1-e^{-100})}$ *is a constant.*

(a) *(5 points) Explain why the population of Elcric is* $\int_0^{10} 2\pi r C e^{-r^2} dr$ *(by explaining where each part of the integral comes from).*

Solution. The area of an annulus (thick circle) of inner radius r and outer radius $r + \Delta r$ has area

$$\Delta A = \pi(r + \Delta r)^2 - \pi r^2 \approx 2\pi r \Delta r,$$

where the difference is on the order of $(\Delta r)^2$, negligibly small. The number of people in this annulus is therefore

$$\Delta P \approx \rho(r)\Delta A \approx Ce^{-r^2}(2\pi r)\Delta r.$$

We can use a Riemann sum for any finite value of Δr to approximate the population. In the limit as $\Delta r \to 0$, we get an integral

$$P = \int dP = \int_0^{10} 2\pi C r e^{-r^2}\, dr.$$

□

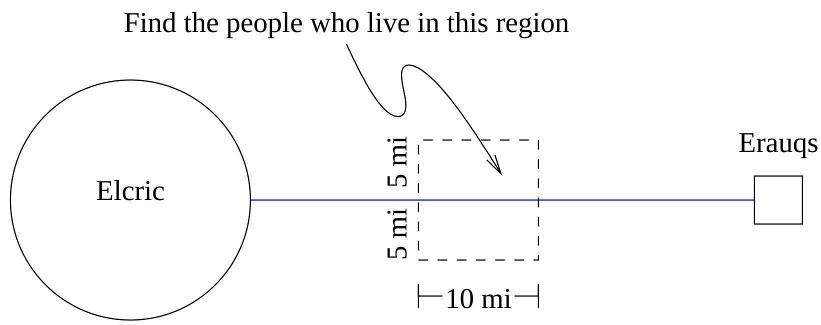
(b) *(5 points) Calculate the population of Elcric.*

Solution. We employ the substitution $u = r^2$, $du = 2r\,dr$. Thus

$$\begin{aligned} P &= C\pi \int_0^{100} e^{-u},\, du \\ &= C\pi \left[-e^{-u}\right]_0^{100} = C\pi(1 - e^{-100}) = 8.3 \times 10^7. \end{aligned}$$

□

(c) *(9 points) Along a 10 mile stretch, the river running from Erauqs to Elcric runs straight, and the population density at a point x miles away from the river is $\rho(x) = 10(5 - x)$, as long as $x \leq 5$.*



How many people live within 5 miles of either side of this 10 mile straight stretch of river?

Solution. The number of people who live in a strip of width dx, parallel to the river and x away from it

$$dP = \rho\, dA = 10|5 - x|(10)\, dx = 100|5 - x|\, dx.$$

Hence

$$\begin{aligned} P &= \int_{-5}^{5} 100|5 - x|\, dx = 200 \int_{0}^{5} (5 - x)\, dx \\ &= 200 \left[5x - \frac{x^2}{2} \right]_0^5 = 2500. \end{aligned}$$

□

______ / 19

9. (15 Points) *Solve the initial-value problem*

$$y' = 2x(1 + y^2); \qquad y(0) = 1.$$

Solution. We separate the variables:

$$\frac{dy}{1 + y^2} = 2x\, dx$$
$$\implies \arctan y = x^2 + C.$$

Plugging in the initial conditions, we get $C = \arctan 1 = \frac{\pi}{4}$. So

$$y = \tan\left(x^2 + \tfrac{\pi}{4}\right).$$

□

______ / 15

10. (10 Points) *Crimsonium (Ha on the periodic table) is a very rare radioactive element. Upon enrolling, Fran the Firstyear receives a 100g sample of pure Crimsonium. Upon graduation four years later, however, the amount of Crimsonium in the sample turns out to be only 30g, while the rest has decayed into worthless Bulldogium.*

(a) *What is the half-life of Crimsonium?*

Solution. We know that Crimsonium decays exponentially. Let its rate of decay be k. Then since

$$30 = 100e^{-k\cdot 4},$$

we have

$$k = \frac{1}{4}\ln\left(\frac{10}{3}\right).$$

Now the half-life $t_{1/2}$ is the number such that

$$1 = 2e^{-k\cdot t_{1/2}},$$

so

$$t_{1/2} = \frac{-\ln\left(\frac{1}{2}\right)}{k} = \frac{4\ln 2}{\ln 10 - \ln 3}.$$

□

(b) *How much Crimsonium will be left for Fran's tenth-year reunion (Note: this is ten years after* graduation*)?*

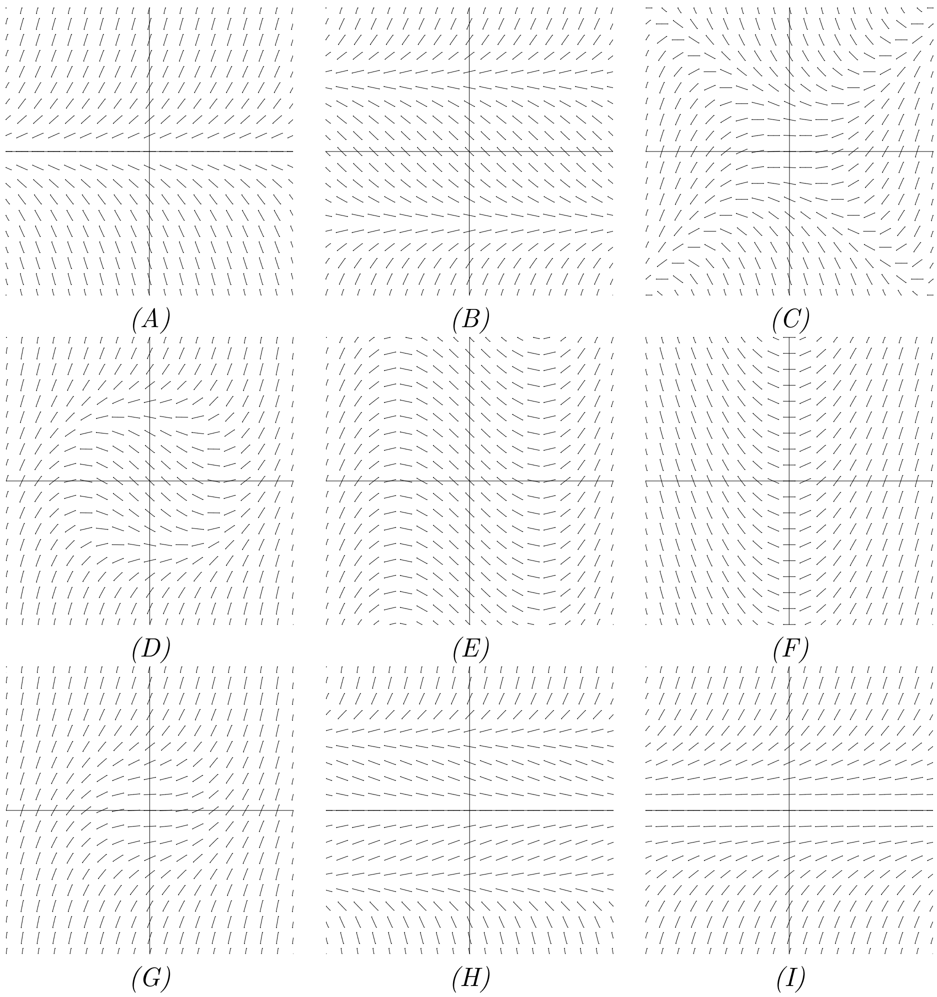
Solution. 14 years after matriculation Fran will have

$$100e^{-k\cdot 14} = 100e^{-\frac{14}{4}\ln\left(\frac{10}{3}\right)} = 100\left(\frac{10}{3}\right)^{7/2}$$

grams of Crimsonium. □

_____ / 10

11. (10 Points) *Each of the five differential equations below can correspond to only one of the nine direction fields. Determine the matches. (The bounds on the plots are $-2 \le x \le 2$, $-2 \le y \le 2$, but you don't need that information to do the problem).*



______ *(i)* $y' = x^2 - 1$

Solution. Because $y' = 0$ when $x = \pm 1$, we should see horizontal tangents (slopes of zero) along these two vertical lines. This is evident in (E). □

______ *(ii)* $y' = y^2 - 1$

Solution. This time, $y' = 0$ when $y = \pm 1$, so the horizontal tangents are along these two horizontal lines. This happens in (B). □

______ *(iii)* $y' = x^2 - y^2$

Solution. Here the horizontal tangents happens when $y = \pm x$, so two lines through the origin. We can see this in (C). □

______ *(iv)* $y' = x^2 + y^2$

Solution. The only zeroes are when $x = y = 0$. This only happens in (G). □

______ *(v)* $y' = x^2 + y^2 - 1$

Solution. The horizontal tangents occur when $x^2 + y^2 = 1$, i.e., on the unit circle. This resembles (D). □

______ / 10

12. (14 Points) *There are two species on Planet Wigglesworth: students and an amazing, living species of pizza slices. Populations of the two species evolve over time according to a predator-prey model:*

$$\frac{dx}{dt} = 0.4x - 0.002xy;$$
$$\frac{dy}{dt} = -0.2y + 0.000008xy.$$

(a) *(3 points) Which of the variables, x or y, represents the student population and which the pizza slice population? (You may assume that students eat pizza slices and not the other way around!)*

Solution. Interactions between x and y increase y and decrease x, so we would assume that x represents slices and y represents students. □

(b) *(5 points) Find the equilibrium points and explain their significance.*

Solution. The equilibria occur when

$$0 = 0.4x - 0.002xy;$$
$$0 = -0.2y + 0.000008xy.$$

We can factor x out of the first and y out of the second, so $(x, y) = (0, 0)$ is an equilibrium point. Not a very interesting one, however; it just says that if there are neither students nor pizza on the planet there will never be any. So if we are not at this point, we may cancel the x from the first and y from the second equation. This yields linear equations for each and we get

$$x = 2500;$$
$$y = 200.$$

This is a nontrivial equilibrium. □

(c) *(3 points) Find an expression for $\frac{dy}{dx}$.*

Solution. We can use the chain rule and get

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{0.4x - 0.002xy}{-0.2y + 0.000008xy}.$$

□

(d) *(3 points) What happens to y in the absence of x?*

Solution. With no x the equation for $\frac{dy}{dt}$ becomes $y' = -0.2y$, which indicates an exponential dying-out (decay) in student population. □

______ / 14

13. (15 Points) *Label each of the following statements as true (**T**) or false (**F**). If the statement is true, explain why. If the statement is false, explain why or give an example that disproves the statement.*

______ *(i) The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges for all positive* p.

Solution. **False.** The p-series does not converge when $0 < p \leq 1$. □

______ *(ii)* $\sum_{n=1}^{\infty} \binom{3/2}{n} 4^{-n} = \frac{1}{2}$.

Solution. **False.** This is a binomial series which converges to $(1 + \frac{1}{4})^{3/2} = \frac{5\sqrt{5}}{8}$. □

______ *(iii)* $\int_{1/2}^{1} \ln x \, dx = \int_{1/2}^{1} e^y \, dy$

Solution. **False.** The left-hand side is negative, but the right-hand side is positive! □

______ *(iv) Suppose the birth rate of a certain country is 3%, the death rate 2% and that 5000 people move out of this country per year (there are no immigrants). If the population of this country is* $P(t)$ *at year* t*, then the differential equation for* $P(t)$ *that describes the above situation is*

$$\frac{dP}{dt} = 0.03P - 0.02P - 5000t.$$

Solution. **False.** The constant rate of emigration should be represented by a -5000, with no t factor. □

______ *(v) The only equilibrium (i.e., constant) solutions to the differential equation* $y' = y(y^2 - 1)$ *are* 0 *and* 1.

Solution. **False.** $y = -1$ is also an equilibrium solution. □

______ / 15