Name: ________________________________

Final Examination

Mathematics 1b

May 21, 2004

Problem	Points	Score
1	10	
2	6	
3	10	
4	9	
5	7	
6	10	
7	5	
8	9	
9	6	
10	12	
11	6	
12	10	
Total	100	

Please show all your work on this exam paper. Unless otherwise indicated, you must show your work and clearly indicate your line of reasoning in order to get full credit. If you have work on the back of a page, indicate that on the exam cover.

Please circle your section.

MWF 10	Weiyang Qiu	MWF 11	Alberto DeSole	MWF 12	Alina Marian
TTH 10	Robin Gottlieb	TTH 11:30	Robin Gottlieb		

1. (10 points)

   (a) In honor of graduating seniors, a brightly colored celebratory pole will be erected in the atrium in the Science Center. The pole is π meters high and has circular cross sections of varying radii; the radius at a height of h meters is given by $r(h) = \sqrt{\frac{1}{10}h + \frac{1}{2}}$ meters. The pole's density is greatest at ground level and is given by $\rho(h) = \cos h + 2$ kilograms per cubic meter. Write an integral that gives the mass of the pole. You need not evaluate the integral.

   Mass of the pole: ______________________________

   (b) A giant spherical fishtank at an aquarium has a radius of 10 feet. It is filled with water to a height of 17 feet. The water is populated by small fish, with a population density of $\rho(y) = 1 + 0.5y$ fish per cubic foot, where y is the elevation from the bottom of the tank. What integral gives the total number of fish in the tank? You need not evaluate the integral.

number of fish in the tank: ______________________________

2. (6 points)

(a) The area under the curve $y = e^{-x^2}$ from $x = 0$ to $x = b$ is rotated about the y-axis. Write an integral that gives the volume generated.

(b) Now show that the volume of the figure obtained by rotating about the y-axis the whole area under $y = e^{-x^2}$ for $x > 0$ is finite. Compute this volume.

3. (10 points)

Determine the interval of convergence of the following power series. If the interval has endpoints, determine whether or not those endpoints are included in the interval of convergence. Explain your reasoning clearly and completely.

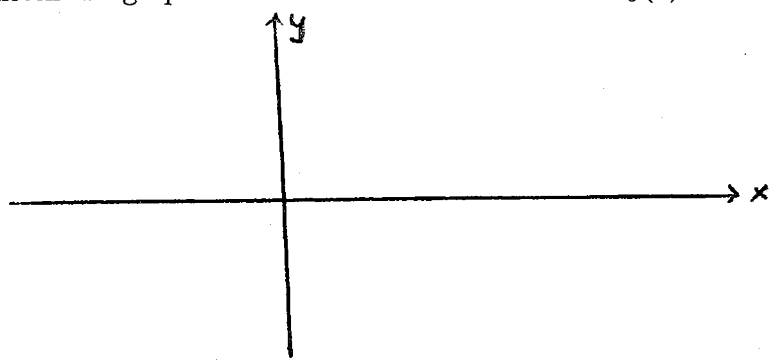
(a) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-4)^n}{\ln n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n(n+4)n^4}$

6. (10 points)

(a) Consider the following differential equation: $\frac{dy}{dx} = 9 - y^2$.

i. Sketch the graph of the solution with initial data $y(0) = 0$.



ii. For the above solution, evaluate the following limits:

$\lim_{x \to \infty} y(x) =$ ____________ $\lim_{x \to -\infty} y(x) =$ ____________

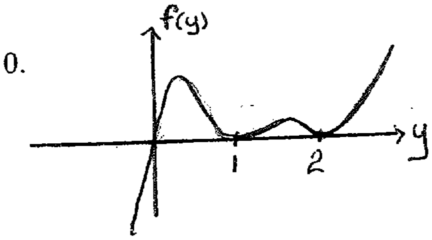
iii. Does the differential equation $\frac{dy}{dx} = 9 - y^2$ have any stable equilibrium solutions? If so, list them. __________

(b) Consider the differential equation

$$\frac{dy}{dt} = f(y),$$

where $f(y)$ is a given function with the following characteristics.

- $f(y) = 0$ only at $y = 0$, $y = 1$, and $y = 2$
- $f(y)$ is negative for $y < 0$ and $f(y) \geq 0$ for $y > 0$.

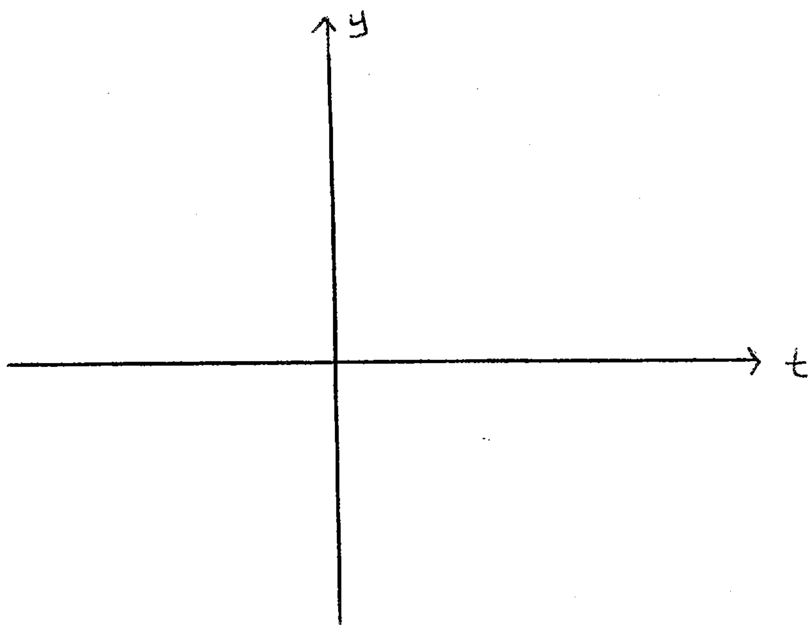


On the set of axes below, sketch the solutions passing through the following points:

i. $y(0) = -2$ ii. $y(1) = \frac{1}{2}$

iii $y(-3) = \frac{5}{4}$ iv. $y(4) = 3$

For each solution, indicate clearly the asymptotic behavior of y as t grows or becomes negative.

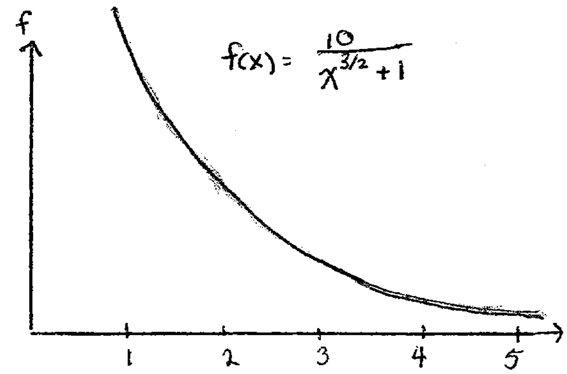


7. (5 points)

Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \int_{2n}^{2n+1} \frac{10}{x^{3/2}+1}\,dx$.

Does the series converge or diverge? Explain your reasoning clearly.

For your convenience we have included a sketch of $f(x) = \frac{10}{x^{3/2}+1}$. You may find a graphical approach useful, but are under no legal obligation to refer to it at all!



8. (9 points) In this problem we consider the motion of an object of mass $m = 1\text{kg}$ attached to a spring. Let $x(t)$ be the position of the mass at time t, where x is measured in meters and t in seconds. We know that $x(t)$ is a solution to a differential equation of the form

$$x'' + bx' + cx = 0.$$

(a) Suppose the mass attached to spring A satisfies the equation

$$x'' + 2x' + 2x = 0.$$

Find the position $x(t)$ of the mass knowing that the initial displacement from equilibrium is 5 and the initial velocity is equal to -5. Find the first time t at which the mass returns to its equilibrium position (i.e. for which $x(t) = 0$) ?

$x(t) =$ ____

The first t for which $x(t) = 0$: ________

(b) Suppose the mass attached to spring B operates in a **frictionless** system; its motion is periodic with a period of $\frac{1}{2}$ second (meaning $x(t + \frac{1}{2}) = x(t)$ for all $t \geq 0$). This information completely determines the equation of motion. Find b and c in the differential equation above.

9. (6 points)

Let $M(t)$ be the amount of money (in hundreds of dollars) John has in his bank account in a given year t. Suppose his money earns interest, with an annual interest rate of 10% compounded continuously. John's original bank balance, at $t = 0$, was \$10,000, ($M = 100$) and every year he deposits (continuously) \$500 in savings. Moreover, John withdraws money from his bank account (continuously) at a rate which increases linearly in time: \$100 the first year, \$200 the second year, and so on.

(a) Model the situation using a differential equation involving $\frac{dM}{dt}$.

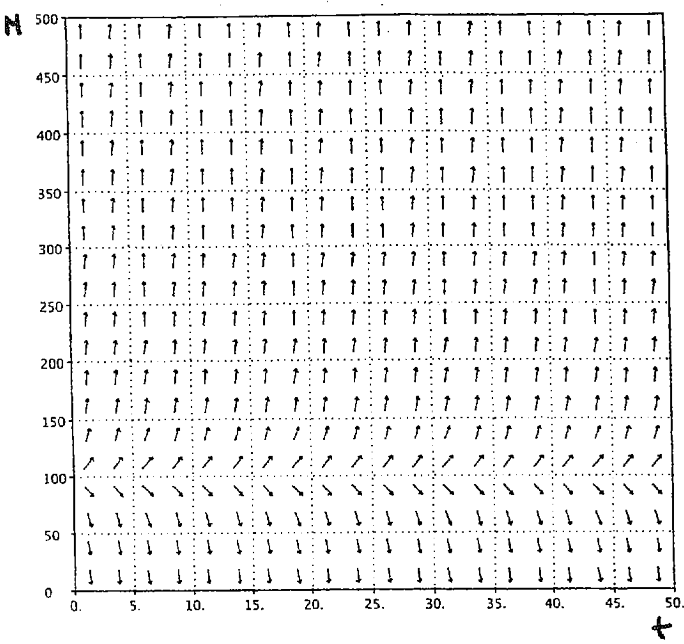
(b) On the next page you will find slope fields. Which one of them is the slope field for the differential equation in part (a)? You need not explain your answer.

For the next two questions it is not necessary to solve the differential equation in order to answer the question.

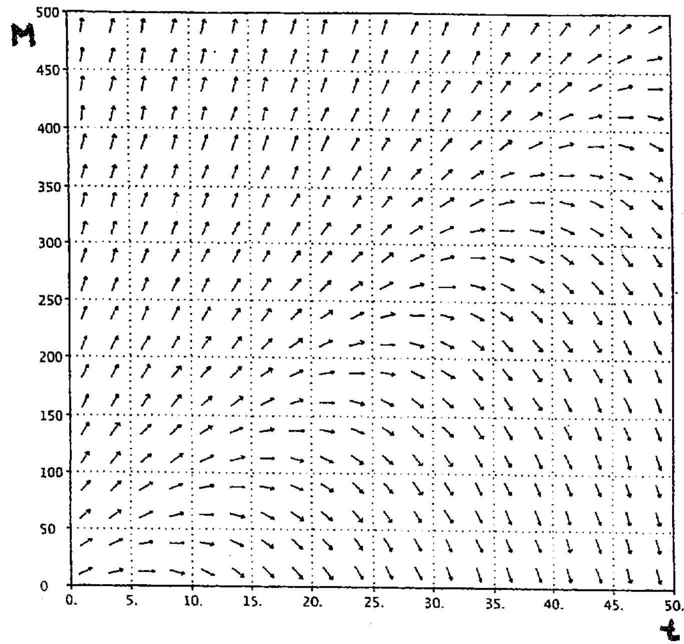
(c) In time, will John become rich or poor?

(d) If John started with a balance of \$3000, ($M = 30$), would he become rich or poor?

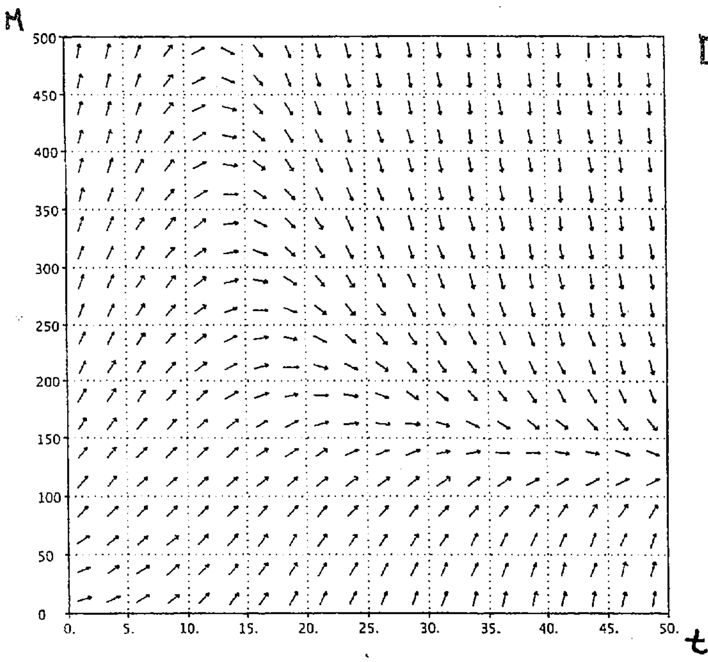
A


M
t
0 50 100 150 200 250 300 350 400 450 500
0. 5. 10. 15. 20. 25. 30. 35. 40. 45. 50.

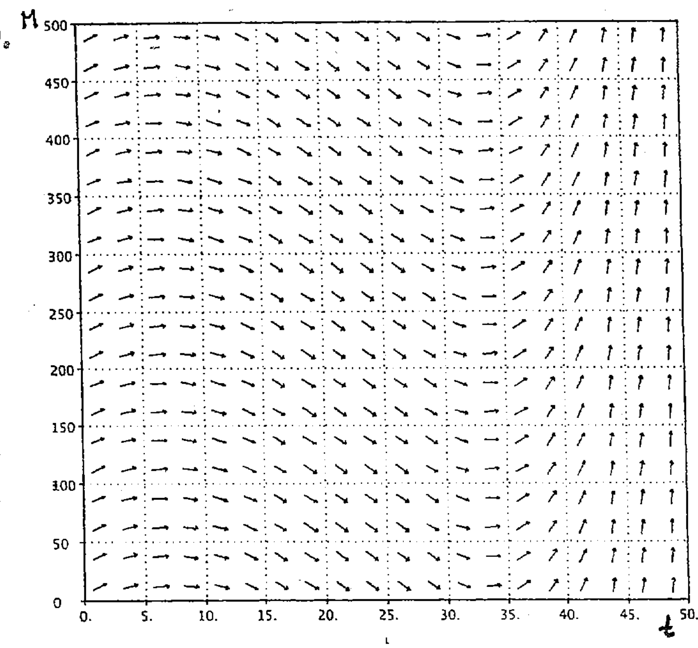
B.


M
t
0 50 100 150 200 250 300 350 400 450 500
0. 5. 10. 15. 20. 25. 30. 35. 40. 45. 50.

C.


M
t
0 50 100 150 200 250 300 350 400 450 500
0. 5. 10. 15. 20. 25. 30. 35. 40. 45. 50.

D.


M
t
0 50 100 150 200 250 300 350 400 450 500
0. 5. 10. 15. 20. 25. 30. 35. 40. 45. 50.

10. (12 points) Let $x = x(t)$ be the number of animals of species A (given in hundreds). Let $y = y(t)$ be the number of animals of species B (given in hundreds). The two species of animals interact as follows:

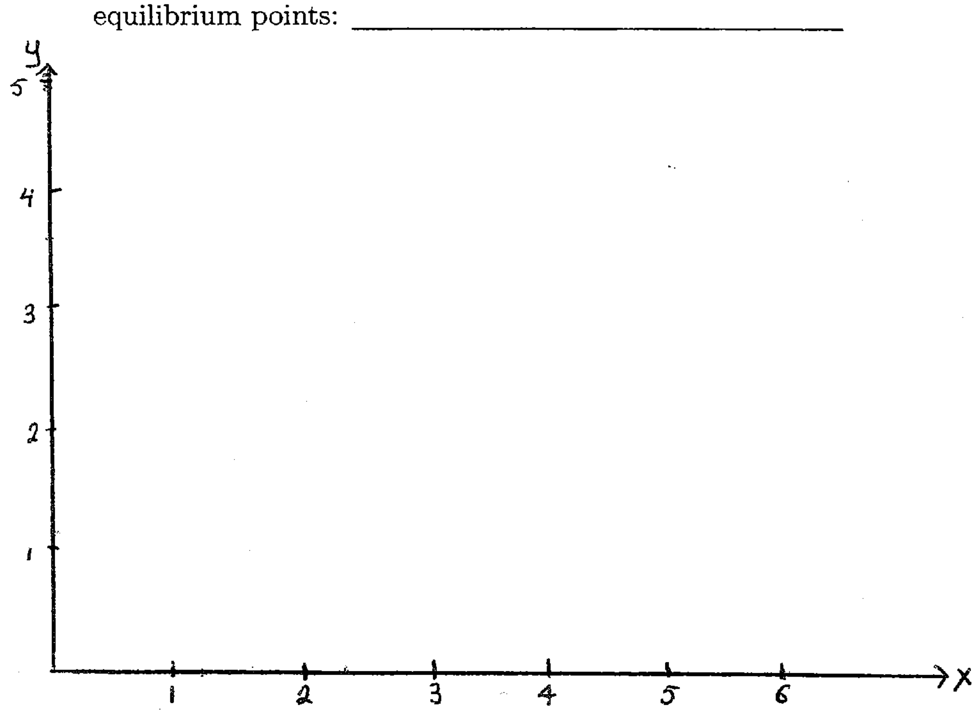
$$\begin{cases} \dfrac{dx}{dt} = 0.1x(4 - x - y). \\ \\ \dfrac{dy}{dt} = 0.1y(3 - x). \end{cases}$$

(a) Describe the relationship between the two species (symbiotic (the presence of each contributing to the welfare of the other), competitive, or predator and prey.)

(b) Describe what happens to each species in the long run in the absence of the other species.

(c) In the first quadrant of the xy-plane, sketch the null clines and label all the equilibrium points. Each null cline should have hash marks with arrows on them indicating the direction that the solution trajectories will take when they cross or travel on the null cline. The null clines will partition the first quadrant into regions; in each region draw an arrow indicating the direction of trajectories in the region.

equilibrium points: ______________________



This problem is continued on the next page.

(d) Suppose $x(0) = 2.9$ and $y(0) = 0.5$. (There are 290 animals of species A and 50 of species B.) What will happen in the short run? Will x increase, or decrease? Will y increase, or decrease? What will happen to both x and y in the long run?

(e) Suppose $x(0) = 2.9$ and $y(0) = 2.9$. (There are 290 animals of species A and 290 of species B.) What will happen in the short run? Will x increase, or decrease? Will y increase, or decrease? What will happen to both x and y in the long run?

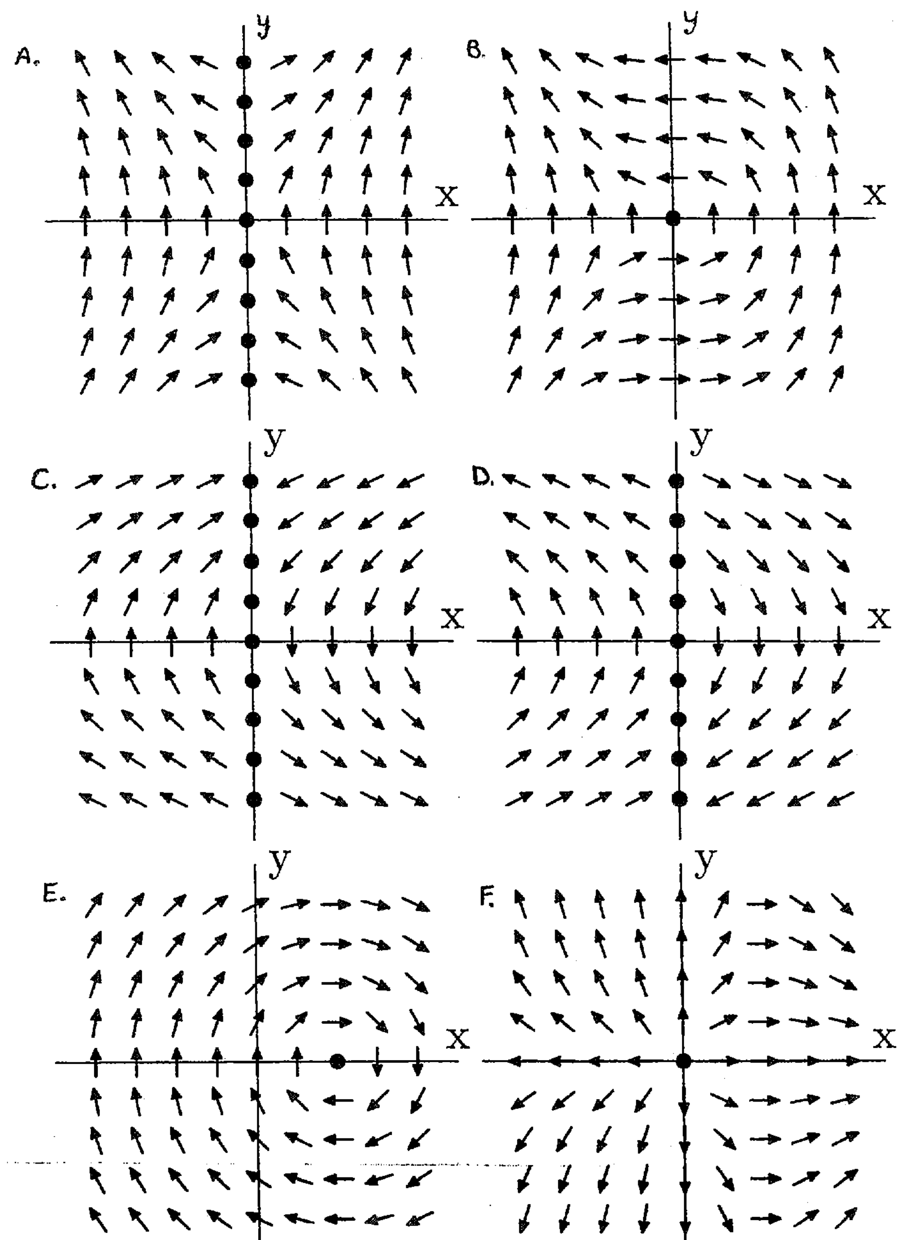
(f) Look back at your answers to the previous parts of this problem and make sure that your answers and your phase plane diagram agree. If they do not agree but you are unable to find the error, please note that below.

11. (6 points) Match each system of differential equations, system I, II, and III, with the phase plane diagram to which it best corresponds. Each system corresponds to exactly one of the phase plane portraits below. (Dots in the phase plane respresent equilibrium points.) You need not explain your work.

System I: $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 1 - x \end{cases}$ System II: $\begin{cases} \dfrac{dx}{dt} = xy \\ \dfrac{dy}{dt} = 2x^2 \end{cases}$ System III: $\begin{cases} \dfrac{dx}{dt} = -xy \\ \dfrac{dy}{dt} = -x \end{cases}$

Answers:

System I ______ System II _______ System III _______



12. (10 points)

Solve the following differential equations for y:

(a) $\frac{dy}{dx} = \frac{x \ln x}{y^2}$ where $y(1) = 1$.

(b) $\frac{dy}{dx} - \frac{3y}{x+1} - x = 0$

Find the general solution and express y as a function of x.