Answers for January 25, 1999 Math 1B Final Exam

(1) (i)(a) converges (... by ratio test $\lim_{n\to\infty} \left(\dfrac{\left(10^{(10^{10})}\right)^{n+1}}{(2n+3)!} \cdot \dfrac{(2n+1)!}{\left(10^{(10^{10})}\right)^{n}} \right)$

$$= \lim_{n\to\infty} \left(\frac{10^{(10^{10})}}{(2n+3)(2n+2)} \right) = 0 < 1 \Bigg)$$

(b) diverges (... divergence test $\lim_{n\to\infty} \left(30 - \frac{1}{n^2}\right) = 30 \neq 0$)

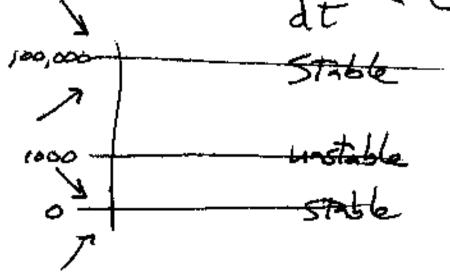
(c) converges (... alt. series test $\lim_{n\to\infty} \dfrac{1}{\sqrt{n^2+10}} = 0$ and abs. value of terms decrease)

(ii) Converges ... compare $\dfrac{k}{k^3+1} = \dfrac{1}{k^2 + 1/k} < \dfrac{1}{k^2}$ (for $k \geq 1$)

and $\sum_{k=1} \dfrac{1}{k^2}$ converges (p-series $p = 2 > 1$)

(2) (a) Equilibrium solutions are at $P = 0$, 1000, and $100{,}000$

check $\dfrac{dP}{dt} < 0$ for $0 < P < 1000$ etc..



$\Rightarrow$ $P = 0$ is a stable equilibrium as is $P = 100{,}000$

$P = 1000$ is an unstable equilibrium

(b) Yes, for $P < 1000$ the population will decrease to zero over time

(c) No, all population values above 100,000 decrease over time back to 100,000

(3) (a) Surface area $= \displaystyle\int_{-r}^{r} 2\pi\, f(x) \sqrt{1 + (f'(x))^2}\, dx$

where $f(x) = \sqrt{r^2 - x^2} = (r^2 - x^2)^{1/2}$

and $f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2} \cdot (-2x)$

(3) (a) continued so $(f'(x))^2 = x^2(r^2-x^2)^{-1} = \frac{x^2}{r^2-x^2}$

and surface area $= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$

$$= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2+x^2}\,dx = \int_{-r}^{r} 2\pi r\,dx$$

$$= 2\pi r x\Big|_{-r}^{r} = 2\pi r^2 - (-2\pi r^2) = 4\pi r^2$$

(b) again $f(x) = \sqrt{r^2-x^2}$

now volume $= \int_{-r}^{r} \pi f(x)^2\,dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx$

$$= \int_{-r}^{r} \pi r^2\,dx - \int_{-r}^{r} \pi x^2\,dx = \pi r^2 x\Big|_{-r}^{r} - \frac{\pi}{3}x^3\Big|_{-r}^{r}$$

$$= \pi r^3 - (-\pi r^3) - \left[\frac{\pi r^3}{3} - \left(-\frac{\pi r^3}{3}\right)\right]$$

$$= 2\pi r^3 - \frac{2}{3}\pi r^3$$

$$= \frac{4}{3}\pi r^3$$

(4) (a) by parts $u = \ln x$ $\quad dv = dx$
$du = \frac{1}{x}dx$ $\quad v = x$

$$\int \ln x\,dx = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - x + C$$

(b) $\int \frac{dx}{x^3-x} = \int \frac{dx}{x(x^2-1)} = \int \frac{dx}{x(x+1)(x-1)}$

$$\frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

or $1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$

sub $x=0 \Rightarrow A=-1$, $x=-1 \Rightarrow B=\frac{1}{2}$, $x=1 \Rightarrow C=\frac{1}{2}$

so $\int \frac{dx}{x^3-x} = \int \frac{-1}{x}dx + \int \frac{1}{2(x+1)}\,dx + \int \frac{1}{2(x-1)}\,dx$

$$= -\ln|x| + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C$$

4 (c) sub $u = \sin x$ $\quad du = \cos x \, dx$

$$= \int_{x=0}^{x=\pi/2} u^2 \, du = \frac{u^3}{3} \Big|_{x=0}^{x=\pi/2} = \frac{(\sin x)^3}{3} \Big|_0^{\pi/2} = \frac{1}{3}$$

(d) sub $u = 1 + x^3$ $\quad du = 3x^2 dx$ $\quad x^3 = u - 1$

$$= \int x^3 \sqrt{1+x^3} \, 3x^2 \, dx = \int (u-1)\sqrt{u} \cdot du = \int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + c = \frac{2}{5}\left(1+x^3\right)^{5/2} - \frac{2}{3}\left(1+x^3\right)^{3/2} + c$$

5 (a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

so $\sin x + \cos x = 1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

(b) $\sin\left(\frac{1}{10}\right) + \cos\left(\frac{1}{10}\right) = 1 + \left(\frac{1}{10}\right) - \frac{\left(\frac{1}{10}\right)^2}{2} - \frac{\left(\frac{1}{10}\right)^3}{3!} + \frac{\left(\frac{1}{10}\right)^4}{4!} + \cdots$

(c) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

$\vdots$

$f^{(5)}(x) = \cos x - \sin x$ $\quad$ so $\left|f^{(5)}(x)\right| \leq 2$ for all x

since $|\cos x| \leq 1$

$|\sin x| \leq 2$

(d) don't need to know how to do this for Fall '99

6 (a) $x \frac{dy}{dx} + 5y = x^2$ or $\frac{dy}{dx} + \frac{5}{x} y = x$ $\quad$ integrating factor

is $e^{\int 5/x \, dx} = e^{5 \ln x} = e^{\ln(x^5)} = x^5$

so $x^5 \cdot y = \int x^5 \cdot x \, dx = \frac{x^7}{7} + c$, so $y = \frac{x^2}{7} + \frac{c}{x^5}$

(6) continued

(b) if $y(x) = \frac{x^2}{7} + \frac{c}{x^5}$ $\quad \frac{dy}{dx} = \frac{2}{7}x - \frac{5c}{x^6}$

so $5y + xy' = 5\left(\frac{x^2}{7} + \frac{c}{x^5}\right) + x\left(\frac{2}{7}x - \frac{5c}{x^6}\right)$

$$= \frac{5}{7}x^2 + \cancel{\frac{5c}{x^5}} + \frac{2}{7}x^2 - \cancel{\frac{5c}{x^5}} = \frac{7}{7}x^2 = x^2 \checkmark$$

(7) once the pail has been raised x feet, the weight of the remaining chain is $(30-x)\cdot 2$, plus the weight of the bucket, which is 100 pounds

so $\text{Work} = \int_0^{30} \left(100 + (60-2x)\right) dx = \int_0^{30} (160-2x)\, dx$

$$= \left(160x - x^2\right)\Big|_0^{30}$$

$$= 4800 - 900 = 3900 \text{ ft. lbs}$$

(8) (a) Rate in − Rate out $= \frac{dc}{dt}$

Rate in $= 2\,\frac{\text{gal}}{\text{min}} \cdot 4\,\frac{\text{mg}}{\text{gal}} = 8\,\frac{\text{mg}}{\text{min}}$

Rate out $= \left(-\frac{2}{500}\right)\cdot c$ $\qquad$ and $c(0) = 1500$

so $\frac{dc}{dt} = 8 - \frac{1}{250}c$ $\qquad$ or $\frac{dc}{dt} + \frac{1}{250}c = 8$

(b) integrating factor $e^{\int \frac{1}{250} dt} = e^{t/250}$

so $c \cdot e^{t/250} = \int 8 \cdot e^{t/250}\, dt = 2000\, e^{t/250} + k$ $\quad$ constant

so $c(t) = 2000 + k \cdot e^{-t/250}$

$c(0) = 1500 = 2000 + k e^0 = 2000 + k \Rightarrow k = -500$

so $c(t) = 2000 - 500\, e^{-t/250}$

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(c) as $t \to \infty$ $C(t) = 2000 - 500e^{-t/250} \to 2000$

(d) here Rate in $= 3\frac{gal}{min} \cdot \frac{4mg}{gal} = 12\frac{mg}{min}$

Rate out $= \left(\frac{2}{\underbrace{500+t}_{\text{total volume after } t \text{ minutes pass}}}\right)$ times $C(t)$

← # gallons out per minute

so $\frac{dC}{dt} = 12 - \left(\frac{2}{500+t}\right)C$

9 (a) $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$ $\quad e^{x^2} = 1 + (x^2) + \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3!} + \cdots$

$= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \cdots = x^2 - \frac{x^6}{3!} + \cdots$

So $e^{x^2} - \sin(x^2) - 1 = \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots\right) - \left(x^2 - \frac{x^6}{3!} + \cdots\right) - 1$

$= \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^6}{3!} + \cdots = \frac{x^4}{2} + \frac{x^6}{3} + \cdots$

(b)(i) since $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

$\frac{1}{1+4x} = \frac{1}{1-(-4x)} = 1 + (-4x) + (-4x)^2 + \cdots = \sum_{n=0}^{\infty} (-4)^n x^n$

(ii) ratio test: $\lim_{n\to\infty} \left|\frac{4^{n+1}x^{n+1}}{4^n x^n}\right| = |4x| < 1$, so $-1 < 4x < 1$

$-\frac{1}{4} < x < \frac{1}{4}$

check endpoints

when $x = -\frac{1}{4}$ get $\sum_{n=0}^{\infty} (-4)^n \left(-\frac{1}{4}\right)^n = \sum_{n=0}^{\infty} 1^n$ diverges

$x = \frac{1}{4}$ ... $\sum_{n=0}^{\infty} (-4)^n \left(\frac{1}{4}\right)^n = \sum_{n=0}^{\infty} (-1)^n$ diverges

so interval of convergence is $-\frac{1}{4} < x < \frac{1}{4}$ radius is $\frac{1}{4}$

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$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^3 + 20a_5 x^3 + 30a_6 x^4 + \cdots$$

$$y(0) = 6 \Rightarrow a_0 = 6, \quad y'(0) = 12 \Rightarrow a_1 = 12$$

next $xy = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + a_5 x^6 + \cdots$

so $y'' + xy = 2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 + (20a_5 + a_2)x^3 + (30a_6 + a_3)x^4 + \cdots$

$= x$

so $2a_2 = 0 \Rightarrow a_2 = 0$

$6a_3 + a_0 = 1$, $a_0 = 6$, so $6a_3 + 6 = 1 \Rightarrow a_3 = -\frac{5}{6}$

$12a_4 + a_1 = 0$, $a_1 = 12$, so $12a_4 + 12 = 0 \Rightarrow a_4 = -1$

$20a_5 + a_2 = 0$, so $2a_5 = 0$, $a_5 = 0$

$30a_6 + a_3 = 0$, $a_3 = -\frac{5}{6}$, so $30a_6 - \frac{5}{6} = 0 \Rightarrow a_6 = \frac{1}{36}$

So

$$y = 6 + 12x + 0 \cdot x^2 - \frac{5}{6}x^3 - x^4 + 0 \cdot x^5 + \frac{1}{36}x^6 + \cdots$$

$$= 6 + 12x - \frac{5}{6}x^3 - x^4 + \frac{x^6}{30} + \cdots$$