A solution for problem 8.6.32a in Stewart
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The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) = 0$$

This solution is deliberately long and drawn-out so that you can see each step clearly. The computation is lengthy but not difficult to understand. First, we calculate the derivatives.

$$\begin{aligned} J_1'(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!(n+1)!2^{2n+1}} \\ J_1''(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n-1}}{n!(n+1)!2^{2n+1}} \end{aligned}$$

(We could let the second derivative start at $n = 1$, but I will defer changing indices for now.) Plugging our derivatives into the differential equation, we get

$$\begin{aligned} &x^2 J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) \\ &= x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n-1}}{n!(n+1)!2^{2n+1}} + x \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{n!(n+1)!2^{2n+1}} \\ &\quad + (x^2 - 1) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!(n+1)!2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n+1}}{n!(n+1)!2^{2n+1}} \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{n!(n+1)!2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} \end{aligned}$$

We can combine these four summations into one.

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \Bigg[\frac{(-1)^n (2n+1)(2n) x^{2n+1}}{n!(n+1)!2^{2n+1}} + \frac{(-1)^n (2n+1) x^{2n+1}}{n!(n+1)!2^{2n+1}} \\
&\qquad + \frac{(-1)^n x^{2n+3}}{n!(n+1)!2^{2n+1}} - \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}} \Bigg] \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) + x^2 - 1}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\frac{4n^2 + 4n + x^2}{n!(n+1)!2^{2n+1}} \right] x^{2n+1}
\end{aligned}
$$

It's a little difficult to work with that x^2 term. Let's try to take care of it by separating it.

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} (-1)^n \left[\frac{4n^2 + 4n}{n!(n+1)!2^{2n+1}} + \frac{x^2}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\frac{4n^2 + 4n}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} + \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^2}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \left[\frac{4n^2 + 4n}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} + \sum_{n=0}^{\infty} (-1)^n \left[\frac{x^{2n+3}}{n!(n+1)!2^{2n+1}} \right]
\end{aligned}
$$

We want the x^{2n+3} in the second series to look like x^{2n+1}. We can do this by starting the series at $n = 1$ and subtracting 1 from each n. Furthermore, notice that the first term of the first series is 0, so we can start this series at $n = 1$ as well without having to change anything else.

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 + 4n}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} + \sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{x^{2n+1}}{(n-1)!n!2^{2n-1}} \right] \\
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 + 4n}{n!(n+1)!2^{2n+1}} \right] x^{2n+1} - \sum_{n=1}^{\infty} (-1)^n \left[\frac{x^{2n+1}}{(n-1)!n!2^{2n-1}} \right] \\
&= \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 + 4n}{n!(n+1)!2^{2n+1}} - \frac{1}{(n-1)!n!2^{2n-1}} \right] x^{2n+1}
\end{aligned}
$$

Get a common denominator . . .

$$= \sum_{n=1}^{\infty}(-1)^n\left[\frac{4n^2+4n}{n!(n+1)!2^{2n+1}} - \frac{n(n+1)\cdot 2^2}{n!(n+1)!2^{2n+1}}\right]x^{2n+1}$$
$$= \sum_{n=1}^{\infty}(-1)^n\left[\frac{4n^2+4n}{n!(n+1)!2^{2n+1}} - \frac{4n^2+4n}{n!(n+1)!2^{2n+1}}\right]x^{2n+1}$$
$$= \sum_{n=1}^{\infty}(-1)^n(0)x^{2n+1} = 0 \qquad \blacksquare$$