Chapter 8: Infinite Sequences and Series, Calculus (3e) by Stewart
Important theorems and convergence tests (compiled by Niels Joaquin)

Theorem (Squeeze Theorem for Sequences). If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem (Monotonic Sequence Theorem). Every bounded, monotonic sequence is convergent.

The **geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$, and its sum is $\frac{a}{1-r}$. If $|r| \geq 1$, the geometric series is divergent.

The Test for Divergence. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

The Integral Test. Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

The ***p*-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

The Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series such that $0 \leq b_n \leq a_n \ \forall n \geq n_0$.
(a) If $\sum a_n$ is convergent, then $\sum b_n$ is convergent.
(b) If $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

The Limit Comparison Test. Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

The Alternating Series Test. If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots \qquad (b_n > 0)$$

satisfies (i) $b_{n+1} \le b_n \ \ \forall n \ge 1$ and (ii) $\lim_{n\to\infty} b_n = 0$, then the series is convergent.

Alternating Series Estimation Theorem. For a convergent alternating series, $|R_n| = |s - s_n| \le b_{n+1}$, where the remainder R_n is the error involved in using the partial sum s_n as an approximation of the total sum s.

Theorem. If a series $\sum |a_n|$ is convergent (i.e., if $\sum a_n$ is absolutely convergent), then $\sum a_n$ is convergent.

The Ratio Test.

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is (absolutely) convergent.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive (i.e., use another test!).

Theorem. For a given **power series** $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:
(i) The series converges only when $x = a$.
(ii) The series converges $\forall x$.
(iii) $\exists R > 0$ (the radius of convergence) such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. (Remember to test the endpoints in this case as well.)

Theorem. If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on the interval $(a-R, a+R)$ and

$$\text{(i)} \qquad f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\text{(ii)} \int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in (i) and (ii) are both R.

Theorem. If f has a power series representation (expansion) at a, that is, if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, where $|x-a| < R$, then its coefficients are given by the formula $c_n = \frac{f^{(n)}(a)}{n!}$. Thus,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This series is called the **Taylor series** of the function f centered at a. When $a = 0$, it is a **Maclaurin series**.

Remark. Note that the theorem is a conditional: "*If* f has a power series representation . . ." So there exist functions that are not equal to the sum of their Taylor series. I have omitted two theorems on this topic, regarding the remainder of a Taylor series (see p. 608 of Stewart).

Some common Maclaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad \forall x \in (-1,1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \forall x \in \mathbb{R}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \forall x \in \mathbb{R}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \forall x \in \mathbb{R}$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad \forall x \in [-1,1]$$

The Binomial Series. If $k \in \mathbb{R}$ and $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$ for $n \geq 1$ and $\binom{k}{0} = 1$.