Tests for Convergence of a Series

Original Source: facultyfiles.deanza.edu/gems/bloomroberta/ConvergenceTests.doc

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Test for Divergence

Given any series Σa_n, if $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the series Σa_n is divergent.

Note that given any series Σa_n, if $\lim_{n\to\infty} a_n = 0$, then the series Σa_n may be either convergent or may be divergent and you will need another test to check for convergence of the series.

Special Series

Geometric Series: $\Sigma a_n = \Sigma ar^n$ is convergent for $-1 < r < 1$ and is divergent for $r \geq -1$ or $r \leq 1$.

If an infinite geometric series Σar^n is convergent, then its sum is $a/(1-r) = a\,[1/(1-r)]$

p – Series: $\Sigma a_n = \Sigma\,(1/n)^p$ is convergent for $p>1$ and is divergent for $p \leq 1$.

We do not have a general formula for the sum of a p-Series $\Sigma\,(1/n)^p$

Harmonic Series: $\Sigma\,(1/n)$ is a special case of the p−Series and is divergent because $p = 1$.

Integral Test

Suppose $f(x)$ is a continuous, positive, decreasing function on the interval $[N, \infty)$ and Σa_n is a series with $a_n = f(n)$ for all integers $n \geq N$:

then $\sum_{n=N}^{\infty} a_n$ is convergent if and only if $\int_N^{\infty} f(x)dx$ is convergent .

then $\sum_{n=N}^{\infty} a_n$ is divergent if and only if $\int_N^{\infty} f(x)dx$ is divergent .

Note that while the textbook states this theorem using N = 1, we are actually interested in the long-term eventual behavior of the series and the function. The behavior of a finite number of initial terms will not affect the divergence or convergence of the series, because the sum of those initial terms will always be a finite number.

Limit Comparison Test

Suppose Σa_n and Σb_n are series with positive terms $a_n > 0$ and $b_n > 0$ for all $n \geq N$:

If $\lim_{n\to\infty} \frac{a_n}{b_n} = C$ for some finite number $C > 0$, $C \neq \infty$, then either both series converge or both series diverge.

The comparison tests require that you have a known convergent or known divergent series Σb_n that you can compare on a term by term basis to the terms of Σa_n

Alternating Series Test

Suppose Σa_n is an alternating series $\Sigma a_n = \Sigma(-1)^{n-1} b_n$, consider $\{b_n\}$ where $b_n = |a_n|$:

if $b_{n+1} < b_n$ for all $n \geq N$ and $\lim_{n\to\infty} b_n = 0$, then the series converges.

This test tells you that an alternating series is convergent if the terms are decreasing AND the terms of the series approach 0. Check that all conditions of this test are true!!!
Show that in your work. If the alternating series does not satisfy the conditions of this test, then the series may converge or may diverge.

If the alternating series $\Sigma a_n = \Sigma(-1)^{n-1} b_n$ does not satisfy the condition that $\lim_{n\to\infty} b_n = 0$, then the signed terms a_n of the alternating series will not satisfy $\lim_{n\to\infty} a_n = 0$, so the alternating series must be divergent, by the Test for Divergence.

If the alternating series $\Sigma a_n = \Sigma(-1)^{n-1} b_n$ satisfies the condition $\lim_{n\to\infty} b_n = 0$, but violates the condition that $b_{n+1} < b_n$ for all $n \geq N$ then alternating series Σa_n may be either convergent or divergent and you need to use a different test to determine convergence or divergence.

Absolute-Convergence Test

Suppose Σa_n is an alternating series: If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

If $\sum_{n=1}^{\infty} |a_n|$ converges, the series is "absolutely convergent" which is a stronger condition that convergence. If you can find a test that works for series with positive terms to show that $\sum_{n=1}^{\infty} |a_n|$ converges, then this test tells you that $\sum_{n=1}^{\infty} a_n$ also converges.

If $\sum_{n=1}^{\infty} |a_n|$ diverges, then this test does not give you any information about the convergence or divergence of the alternating series $\sum_{n=1}^{\infty} a_n$ $\sum_{n=1}^{\infty} a_n$. You need to find another test of convergence.

An alternating convergent series $\sum a_n$ **converges absolutely** if $\sum |a_n|$ also converges
An alternating convergent series $\sum a_n$ **converges conditionally** if $\sum |a_n|$ diverges.

Ratio Test: Given a series $\sum_{n=1}^{\infty} a_n$:

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and therefore also convergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test gives no information about convergence or divergence of $\sum_{n=1}^{\infty} a_n$

Note that you can use the ratio test both for series that are not alternating series and for alternating series. The ratio test is useful for series whose terms contain factorials.

Practice Problems – Test for Convergence:

1) $\sum_{n=1}^{\infty} 7n^2e^{-n^3}$

2) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

3) $\sum_{n=1}^{\infty} \frac{1}{n^2+17}$

4) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2+9}$

5) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^2}$

6) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

7) $\sum_{n=0}^{\infty} \frac{n!}{(2n-1)!}$

8) $\sum_{n=1}^{\infty} \frac{n^3+1}{n^5+4n^3+3}$

9) $\sum_{n=0}^{\infty} \frac{2^n \cdot n}{(n+1)^2}$

10) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2+1}}$

11) $\sum_{n=1}^{\infty} \frac{3}{n^3 \cdot 7^n}$

12) $\sum_{n=0}^{\infty} ne^{-n}$

Solutions: See http://archives.math.utk.edu/visual.calculus/6/series.17/index.html
http://archives.math.utk.edu/visual.calculus/6/series.14/index.html
http://archives.math.utk.edu/visual.calculus/6/series.16/index.html
http://archives.math.utk.edu/visual.calculus/6/series.13/index.html

This is a great resource for both series and power series. You may find it helpful to use this while studying: http://archives.math.utk.edu/visual.calculus/6/