

Week 1 Topics — Math 1b

Sections: 7.4,11.3,11.4

Infinite series are just sums of an infinite number of terms. Unlike the case for finite sums, when you try to add the terms of an infinite series, sometimes you get an answer that is infinite. Other times, the usual rules of arithmetic don't appear to apply. This week we began investigating some techniques for determining when you could safely add up all the terms and get something finite. This is the notion of convergence.

Definition 1 Let $\sum_{k=1}^{\infty} u_k$ be a series and $s_n = \sum_{k=1}^n u_k$ be the ***n*-th partial sum**. If $S = \lim_{n \rightarrow \infty} s_n$ exists, then $\sum_{k=1}^{\infty} u_k$ is said to **converge** with **sum** $S = \lim_{n \rightarrow \infty} s_n$. Otherwise, $\sum_{k=1}^{\infty} u_k$ is said to **diverge**.

Some classes of infinite series:

1. **Geometric series:** of the form $\sum_{k=1}^{\infty} ar^k$, where $a \neq 0$.
 - Converges to $a/(1-r)$ when $|r| < 1$.
 - Diverges when $|r| \geq 1$.
2. **p-series:** of the form $\sum_{k=1}^{\infty} 1/k^p$, where $p > 0$.
 - Converges when $p > 1$.
 - Diverges when $0 < p \leq 1$.

Tests for convergence or divergence:

1. See what $\lim_{n \rightarrow \infty} s_n$ is.
2. Divergence Test:
 - If $\lim_{k \rightarrow \infty} u_k \neq 0$, then the series diverges.
 - If $\lim_{k \rightarrow \infty} u_k = 0$, then use another test.

Good examples to keep in mind:

1. **Harmonic series:** $1/1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$
2. **GS** (with $a = 1, r = 1/2$): $1/1 + 1/2 + 1/4 + 1/8 + \dots$

Some questions to consider:

1. What are the advantages/disadvantages of the Divergence Test as compared to looking at $\lim_{n \rightarrow \infty} s_n$?
2. Do we have a formula for what a p -series ($p > 1$) converges to?
3. Find a way to modify $x \cdot \tan(1/x)$ so that you could apply l'Hôpital's Rule to:

$$\lim_{x \rightarrow \infty} x \cdot \tan(1/x)$$

Week 2 Topics — Math 1b

Sections: 11.6,11.7

Infinite series are more general than finite summations. This increase in flexibility means, however, that sometimes it doesn't make sense to assign a "sum" to the infinite series. The tests listed below can help determine when a series satisfies the property of *convergence* — and can therefore be "summed." These tests usually *don't* tell us what the "sum" is.

Note that you are *not* responsible for the root test or the integral test.

Theorem 1 (Divergence) Let $\sum_{k=1}^{\infty} u_k$ be a series.

1. If $\lim_{k \rightarrow \infty} u_k \neq 0$, then the series does not converge.
2. If $\lim_{k \rightarrow \infty} u_k = 0$, then use another test.

Theorem 2 (Comparison) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with positive terms and suppose that $a_l \leq b_l$ for all sufficiently large l . Then

1. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
2. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

Theorem 3 (Alternating Series) Let $\sum_{k=1}^{\infty} (-1)^k a_k$ be an alternating series. The series converges if the following two conditions hold:

1. $a_l > a_{l+1}$ for all sufficiently large l .
2. $\lim_{k \rightarrow \infty} a_k = 0$.

Theorem 4 (Limit Comparison) Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with positive terms. Let $\rho = \lim_{k \rightarrow \infty} a_k/b_k$. If $0 < \rho < \infty$, then either

1. $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, or
2. $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both diverge.

Theorem 5 (Ratio) Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms. Let $\rho = \lim_{k \rightarrow \infty} a_{k+1}/a_k$. Then

1. If $\rho < 1$ then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\rho > 1$ then $\sum_{k=1}^{\infty} a_k$ diverges.
3. If $\rho = 1$ then use another test.

Theorem 6 (Absolute Value) If $\sum_{k=1}^{\infty} |a_k|$ converges, then so does $\sum_{k=1}^{\infty} a_k$.

Theorem 7 (Ratio with Absolute Values) Let $\sum_{k=1}^{\infty} a_k$ be a series with non-zero terms. Let $\rho = \lim_{k \rightarrow \infty} |a_{k+1}|/|a_k|$. Then

1. If $\rho < 1$ then $\sum_{k=1}^{\infty} a_k$ converges.

2. If $\rho > 1$ then $\sum_{k=1}^{\infty} a_k$ diverges.
3. If $\rho = 1$ then use another test.

In order to help in applying these tests, we defined three different classes of series. These classes were defined according to what restrictions we based on the sequences of coefficients. The importance of these classes stems from their help in determining which convergence tests can be applied to a particular series. So don't worry about memorizing the classes if you know when to apply the tests. But thinking about the classes may help you organize your thoughts. The three classes are:

1. Positive: $\sum_k a_k$, each $a_k > 0$.
2. Alternating: $\sum_k (-1)^k a_k$, each $a_k > 0$.
3. Arbitrary: $\sum_k a_k$, each a_k non-zero.

The following table tells us which of the above tests can be used for particular classes of series. The final column gives a tip as to when each test is often useful.

	Positive	Alternating	Arbitrary	When to use
DT	✓	✓	✓	Terms get "big"
CT	✓			Similar to a series you know, polynomials
LCT	✓			Similar to a series you know, polynomials
RT	✓			factorials, k -th powers
AST		✓		Have alternating series
AV		✓	✓	"Positive" form converges
RT with AV	✓	✓	✓	factorials, k -th powers

Some questions to consider:

1. Which tests can be used for series $\sum_{k=1}^{\infty} a_k$ for which some of the a_k might be 0?
2. Give examples of series $\sum_{k=1}^{\infty} a_k$ for which $\rho = 1$ in the ratio test.
3. Suppose $\sum_{k=1}^{\infty} a_k$ is a series and $|a_{k+1}|/|a_k| = 1/6$ for every k . Can I make any conclusions about convergence?
4. Why does the AST let you conclude convergence in some cases whereas the DT never lets you conclude convergence?
5. Convince yourself that the AV Theorem makes sense.
6. Give an example of an *arbitrary* series for which the Comparison Test will fail.
7. Does convergence of $\sum_{k=1}^{\infty} a_k$ tell you anything about convergence of $\sum_{k=6}^{\infty} a_k$? What about their respective sums?

As with all of the topics we'll be doing this semester, the best way to become comfortable with the material is to do lots of problems. I'm happy to suggest extra problems if you feel as though you need more work on something.