SMB Ex. 1, 4, 5, 7 p. 17

1. $y'' + 4y' + 3y = 0$

$r^2 + 4r + 3 = 0 \Rightarrow r = \frac{-4 \pm 2}{2} \begin{matrix} \nearrow -1 \\ \searrow -3 \end{matrix}$

$\therefore \; y(t) = c_1 e^{-3t} + c_2 e^{-t}$

4. $s'' - 7s = 0$

$r^2 - 7 = 0 \Rightarrow r = \pm\sqrt{7}$

$\therefore \; s(t) = c_1 e^{\sqrt{7}t} + c_2 e^{\sqrt{7}t}$

5. $s'' + 7s = 0$

$r^2 + 7 = 0 \Rightarrow r = \pm i\sqrt{7}$

$\therefore \; s(t) = c_1 e^{i\sqrt{7}t} + c_2 e^{-i\sqrt{7}t}$

or

$s(t) = c_1' \cos(\sqrt{7}t) + c_2' \sin(\sqrt{7}t)$

7. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 8x = 0$

$r^2 + 4r + 8 = 0 \Rightarrow r = \frac{-4 \pm i4}{2} \begin{matrix} \nearrow -2+2i \\ \searrow -2-2i \end{matrix}$

$\therefore \; x(t) = c_1 e^{(-2+2i)t} + c_2 e^{(-2-2i)t}$

or, (w/ $\alpha = -2$, $\beta = 2$)

$x(t) = e^{-2t}\left(c_1' \cos(2t) + c_2' \sin(2t)\right)$

SMB Ex. 9, 10 p. 18

9. $y'' + 6y' + 5y = 0$, $y(0) = 1$, $y'(0) = 0$

$r^2 + 6r + 5 = 0 \Rightarrow r = \frac{-6 \pm 4}{2} \begin{matrix} -1 \\ -5 \end{matrix}$

$\therefore y(t) = c_1 e^{-5t} + c_2 e^{-t}$

$y(0) = c_1 + c_2 = 1$

$y'(t) = -5c_1 e^{-5t} - c_2 e^{-t} \Rightarrow y'(0) = -5c_1 - c_2 = 0 \Rightarrow$

$\Rightarrow -4c_1 - (\cancel{c_1 + c_2}^{\,1}) = 0 \Rightarrow -4c_1 = 1 \Rightarrow c_1 = -\frac{1}{4}$

and $c_2 = \frac{5}{4}$

Hence, $y(t) = -\frac{1}{4} e^{-5t} + \frac{5}{4} e^{-t}$

10. $y'' + 6y' + 5y = 0$, $y(0) = 5$, $y'(0) = 5$

using $y(t) = c_1 e^{-5t} + c_2 e^{-t}$ from above:

$y(0) = c_1 + c_2 = 5$

$y'(0) = -4c_1 - (\cancel{c_1 + c_2}^{\,5}) = 5 \Rightarrow -4c_1 = 10 \Rightarrow c_1 = -\frac{5}{2}$

and $c_2 = \frac{15}{2}$

Hence, $y(t) = -\frac{5}{2} e^{-5t} + \frac{15}{2} e^{-t}$

SMB Ex. 11, 12 p. 18

11. $y'' + 6y' + 10y = 0$, $y(0) = 0$, $y'(0) = 2$

$r^2 + 6r + 10 = 0 \Rightarrow r = \frac{-6 \pm 2i}{2} = -3 \pm i$

$\therefore y(t) = c_1 e^{(-3+i)t} + c_2 e^{(-3-i)t}$

$= e^{-3t}(c_1' \cos t + c_2' \sin t)$

$y'(t) = -3e^{-3t}(c_1' \cos t + c_2' \sin t) + e^{-3t}(-c_1' \sin t + c_2' \cos t)$

$y(0) = c_1' = 0$

$y'(0) = -c_1' + c_2' = 2 \Rightarrow c_2' = 2$

Hence, $y(t) = 2e^{-3t} \sin t$

12. $y'' + 6y' + 10y = 0$, $y(0) = 0$, $y'(0) = 0$

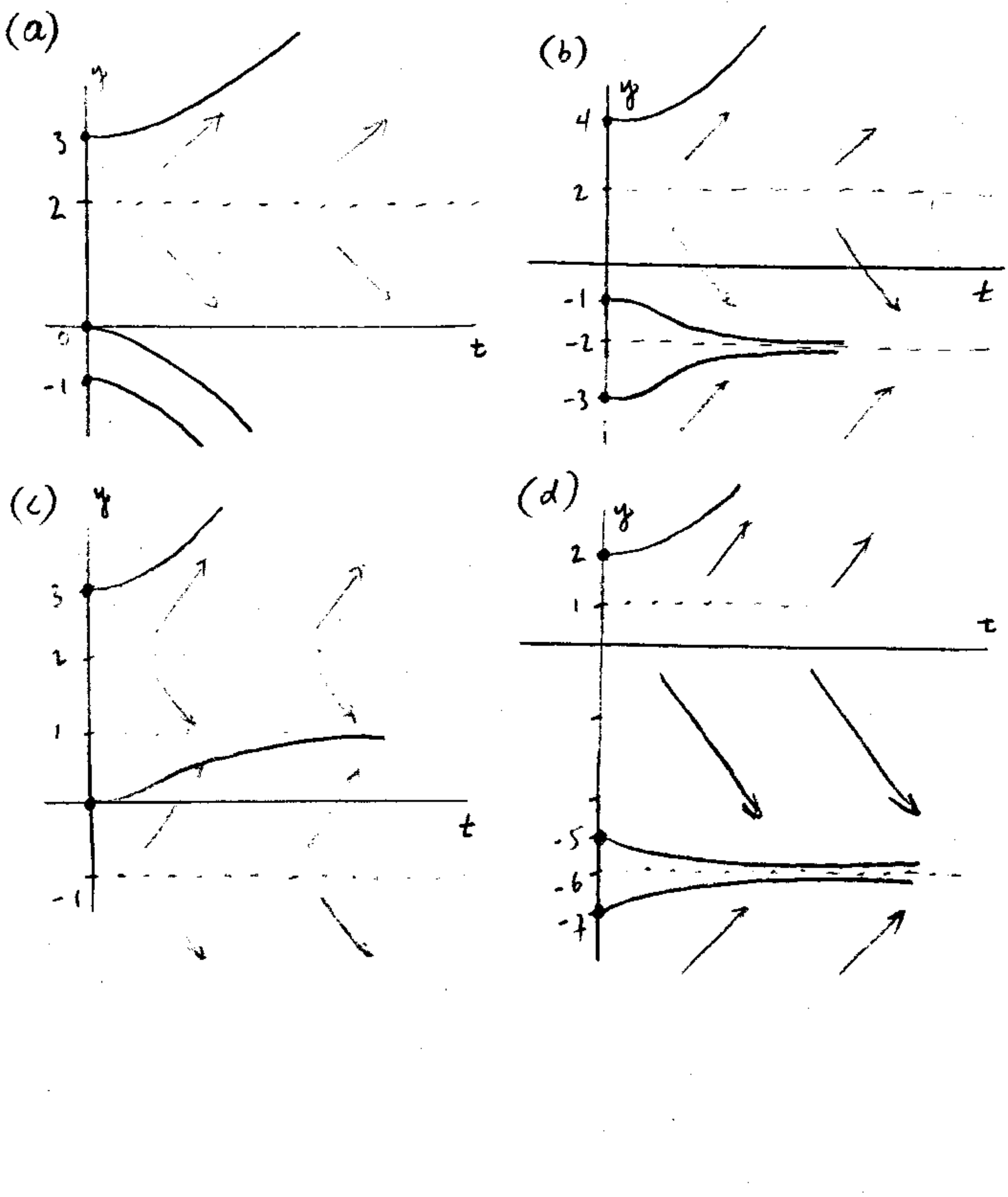
From above,

$y(0) = c_1' = 0$

$y'(0) = -c_1' + c_2' = 0 \Rightarrow c_2' = 0$

Hence $y(t) = 0$

Graphs for SMB Ex. 1 p. 28-29:



SMB Ex. 1 p. 28-29

(a) $y' = 4y - 8 = 4(y-2) = 0 \Rightarrow y = 2$ (equilibrium solution

y	$-\infty$	2	$+\infty$
sign of y'	$-$		$+$

$$y'' = \frac{d}{dt}(y') = \frac{d}{dy} \cdot \frac{dy}{dt} \cdot (y') = \frac{d}{dy}(y') \cdot \frac{dy}{dt}$$

$$= 4 \cdot \frac{dy}{dt} = 4y' = 16(y-2) = 0 \Rightarrow$$

$\Rightarrow y = 2$

y	$-\infty$	2	$+\infty$
sign of y''	$-$		$+$

(b) $y' = y^2 - 4 = (y-2)(y+2) = 0 \Rightarrow y = \pm 2$

y	$-\infty$	-2	2	$+\infty$
sign of y'	$+$	$-$	$+$	

$$y'' = \frac{d}{dy}(y') \cdot y' = 2y \cdot y' = 0 \Rightarrow$$

$\Rightarrow y = -2, 0, 2$

y	$-\infty$	-2	0	2	$+\infty$
sign of y'	$-$	$+$	$-$	$+$	

(c) $y' = (y-1)(y-2)(y+1) = 0 \Rightarrow$

$\Rightarrow y = -1, 1, 2$ (equilibrium points)

y	$-\infty$		-1		1		2		$+\infty$
sign of y'		$-$		$+$		$-$		$+$	

$$y'' = \frac{d}{dy}(y') \cdot \frac{dy}{dt} = \frac{d}{dy}\left[(y^2-1)(y-2)\right] \cdot y' =$$

$$= \frac{d}{dy}\left(y^3 - 2y^2 - y + 2\right) \cdot y' = (3y^2 - 4y - 1) \cdot y' = 0 \Rightarrow$$

$$\Rightarrow y = \frac{4 \pm \sqrt{16+12}}{2}, \pm 1, 2 = 2 \pm \sqrt{7}, \pm 1, 2$$

y	$-\infty$		-1		$2-\sqrt{7}$		1		2		$2+\sqrt{7}$		$+\infty$
sign of y''		$-$		$+$		$-$		$+$		$-$		$+$	

(d) $y' = y^2 + 5y - 6 = 0 \Rightarrow y = \dfrac{-5 \pm \sqrt{24+25}}{2} \Rightarrow$

$\Rightarrow y = \dfrac{-5 \pm 7}{2} = -6, +1$

y	$-\infty$		-6		1		$+\infty$
sign of y'		$+$		$-$		$+$	

$$y'' = \frac{d}{dy}(y') \cdot y' = (2y+5) \cdot y' = 0 \Rightarrow$$

$\Rightarrow y = -6, -5/2, 1$

y	$-\infty$		-6		$-5/2$		1		$+\infty$
sign of y''		$-$		$+$		$-$		$+$	

SMB Ex. 2 p. 29

(a) Need $f(y)$ such that $f(y) = 0 \Rightarrow y = 3$ and such that

y	$-\infty$		3		$+\infty$
sign of y'		$+$		$-$	
sign of y''		$-$		$+$	

Construct $f(y)$ that has a solution $y=3$, a negative slope for $y>3$ and positive otherwise, and a concavity that has the opposite sign of the slope.

One such solution is $f(y) = 3-y$

Verify:

$\dot{y} = f(y) = 3-y = 0 \Rightarrow y = 3$ (equilibrium solution)

$y'' = \frac{d}{dy}(y') \cdot y' = (-1)\cdot y' = -y' = y-3 = 0 \Rightarrow$

$\Rightarrow y = 3$

y	$-\infty$		3		$+\infty$
sign of y'		$+$		$-$	
sign of y''		$-$		$+$	

(b) Conditions that $f(y)$ must satisfy:

$f(3) = 0$

y	$-\infty$		3		$+\infty$
sign of y'		$-$		$+$	
sign of y''		$-$		$+$	

One such solution is $y' = f(y) = y - 3$

(c) Conditions that $f(y)$ must satisfy:

$f(0) = f(2) = 0$

concavity changes at $y = 1 \Rightarrow f'(1) = 0$

y	$-\infty$		0		1		2		$+\infty$
sign of y'		$-$		$+$		$+$		$-$	
sign of y'		$-$		$+$		$-$		$+$	

First, construct a function with the same equilibrium solutions and then manipulate in order to match the rest of the conditions.

One solution therefore is $y' = f(y) = y(2-y)$

(d) Conditions that $f(y)$ must satisfy:

$f(-1) = f(1) = 0$

concavity changes at $y=0 \Rightarrow f'(0) = 0$

y	$-\infty$		-1		0		1		$+\infty$
sign of y'		$+$		$-$		$-$		$+$	
sign of y''		$-$		$+$		$-$		$+$	

We can verify that one such solution is the function we construct when we consider the equilibrium points alone. Hence $y' = f(y) = y^2 - 1$

SMB Ex. 3 p. 29

The equilibria are those points for which $y' = f(y) = 0$, i.e. where our graphs have a slope of zero. Such an equilibrium is stable if our graphs approach this value from both above and below when we are considering a small neighbourhood of points around the equilibrium. Even if only one of the two graphs diverges, then the equilibrium is unstable.

	Equilibrium points	Stable?
(a)	$y=3$	YES
(b)	$y=3$	NO
(c)	$y=0$	NO
	$y=2$	YES
(d)	$y=-1$	YES
	$y=1$	NO

SMB Ex. 4 p. 29

(a) The constant solutions to a differential equation are the solutions for which $\frac{dy}{dt}=0$. Hence, $\frac{dy}{dt}=g(y)=0$

has constant solutions as shown by the graph:

$y = 1, 3, 5$

Also, from the graph we see that:

y	$-\infty$		1		3		5		$+\infty$
sign of dy/dt = sign of $g(y)$		+		−		+		−	

The solution is increasing when $dy/dt > 0$, i.e. for $y \in (-\infty, 1) \cup (3, 5)$

SMB Ex. 1 p. 37

From an arrow representation of the sign of $\frac{dy}{dt}$ we identify equilibrium points as stable if and only if both arrows that are adjacent to the equilibrium we are considering point towards that equilibrium. This is a necessary condition, because only then do we gravitate towards the same equilibrium when assuming slight perturbations in y. Even if only one arrow points away from that equilibrium, a perturbation of y in the direction of the arrow will cause y to diverge.

SMB Ex. 2 p. 37

From fig. 17, we can tell that y_2 and y_4 correspond to inflection points, because

$$y''(y_2) = y''(y_4) = f'(y_2) = f'(y_4) = 0$$

and we know by definition that $y'' = 0$ indicates a point of inflection.

SMB Ex. 3 p. 37

(a) We have assumed that at time t, people are being infected at a rate proportional to the product of those already infected and those who are not, ie $\frac{dP}{dt}$ proportional to $\left(\begin{matrix}\text{people}\\ \text{infected}\end{matrix}\right) \times \left(\begin{matrix}\text{people un-}\\ \text{infected}\end{matrix}\right)$

let P denote the people already infected and N the total population

Then $(N-P)$ is the number of people that have not been infected yet.

Using a proportionality constant k then:

$$\frac{dP}{dt} = k \cdot P \cdot (N-P)$$

$$\int_{P_0}^{P(t)} \frac{dP}{P(N-P)} = \int_0^t k\, dt \Rightarrow \int_{P_0}^{P(t)} \frac{1}{N}\left(\frac{1}{P} + \frac{1}{N-P}\right) dP = kt \Rightarrow$$

$$\Rightarrow \int_{P_0}^{P(t)} \frac{1}{P}\, dP - \int_{P_0}^{P(t)} \frac{1}{N-P}\, d(N-P) = Nk \cdot t \Rightarrow$$

$$\Rightarrow \ln \frac{P(t)}{P_0} - \ln \frac{N-P(t)}{N-P_0} = Nk \cdot t \Rightarrow$$

$$\Rightarrow \ln \left(\frac{P(t)}{N-\cancel{\$}P(t)} \cdot \frac{N-P_0}{P_0}\right) = Nk \cdot t \Rightarrow$$

$$\Rightarrow \frac{P(t)+N-N}{N-P(t)} = \frac{P_0}{N-P_0} \cdot e^{Nk \cdot t} \Rightarrow \frac{P(t)-N}{P(t)-N} + \frac{N}{P(t)-N} = \frac{P_0}{P_0-N} e^{Nkt} \Rightarrow$$

$$\Rightarrow \frac{P(t)-N}{N} = \left[\frac{P_0}{P_0-N} e^{Nkt} - 1\right]^{-1} \Rightarrow P(t) = N\left[\left[\frac{P_0}{P_0-N} e^{Nkt} - 1\right]^{-1} + 1\right] \Rightarrow$$

$$\Rightarrow P(t) = N\left(\frac{P_0-N}{P_0 e^{Nkt} - P_0 + N} + 1\right) = N \cdot \frac{P_0 e^{Nkt}}{P_0(e^{Nkt}-1)+N}$$

(b) We want to maximize $\frac{dP}{dt}$

The first order condition gives us:

$$\frac{d^2P}{dt} = 0 \Rightarrow \frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dP}\left(\frac{dP}{dt}\right) \cdot \frac{dP}{dt} =$$

$$= \frac{d}{dP}\left[kP(N-P)\right] \cdot \left[kP(N-P)\right] =$$

$$= (kN - 2kP) \cdot kP(N-P) = k^2 \cdot P(N-2P)(N-P) = 0 \Rightarrow$$

$\Rightarrow$ maxima and minima occur at

$$P = 0, \; N/2, \; N$$

Plugging back to the rate equation:

$$\left.\frac{dP}{dt}\right|_{P=0} = 0 \qquad \left.\frac{dP}{dA}\right|_{P=N/2} = \frac{kN^2}{4} \qquad \left.\frac{dP}{dA}\right|_{P=N} = 0$$

Hence, the rate at which people are infected is maximum when $P = N/2$, or when half the total population has been infected.

SMB Ex. 4 p. 37

$$\frac{dP}{dt} = .001\,(500-P)\,P$$

$$\frac{d^2P}{dt^2} = \frac{d}{dP}\left(\frac{dP}{dt}\right)\cdot\frac{dP}{dt} = .001^2(500-2P)(500-P)\cdot P$$

set $\frac{dP}{dt} = 0$ to find equilibrium points:

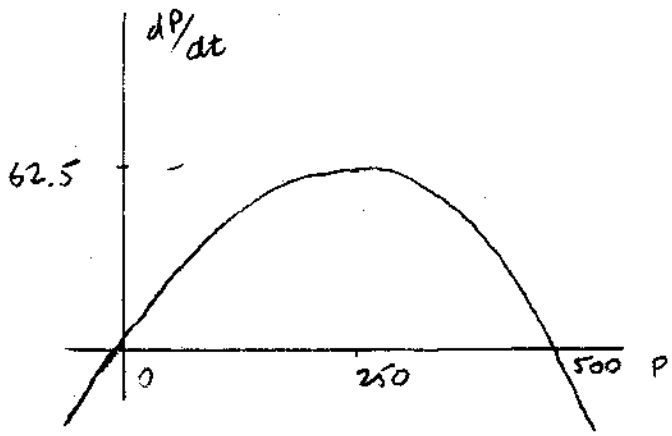
$\frac{dP}{dt} = 0 \Rightarrow P = 0$ or $P = 500$

set $\frac{d^2P}{dt^2} = 0$ to find point of maximum or minimum rate of change

(where the slope of dP/dt becomes zero)

$\frac{d^2P}{dt^2} = 0 \Rightarrow P = 0$ or $P = \frac{500}{2} = 250$ or $P = 500$

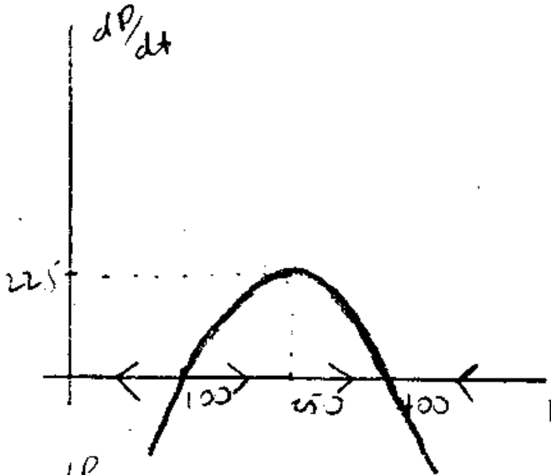
$\frac{dP}{dt}(P = 250) = 62.5$ (max value of dP/dt)

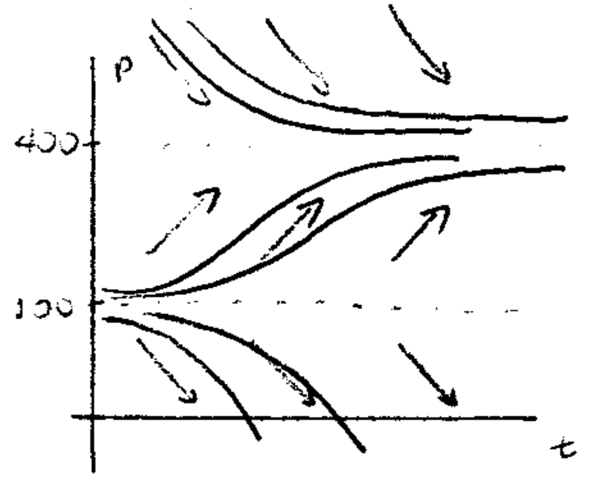


i) Fish caught at a rate of 40/year:

$$\frac{dP}{dt} = .001(500-P)P - 40$$

which translates the original graph of dP/dt by 40 units vertically downwards.





$$\frac{dP}{dt} = 0 \Rightarrow .5P - .001P^2 - 40 = 0 \Rightarrow -P^2 + 500P - 40000 = 0 \Rightarrow$$

$$\Rightarrow P = (500 \pm 300)/2 \rightarrow 100,\ 400$$

P	$-\infty$		100		400		$+\infty$
sign of dP/dt		$-$		$+$		$-$	

$$\frac{d^2P}{dt^2} = \frac{d}{dP}\left(\frac{dP}{dt}\right)\cdot\frac{dP}{dt} = .001^2(500-2P)\left[(500-P)\cdot P - 40\right] =$$

$$= .001^2(500-2P)(-P^2+500P-40) = 0 \Rightarrow$$

$\Rightarrow$ $P = 250$ gives max value of dP/dt

which is $\frac{dP}{dt}(P=250) = 22.5$

The fish population will not be depleted and we have 2 new equilibria, an unstable one at $P=100$ and a stable one at $P=400$

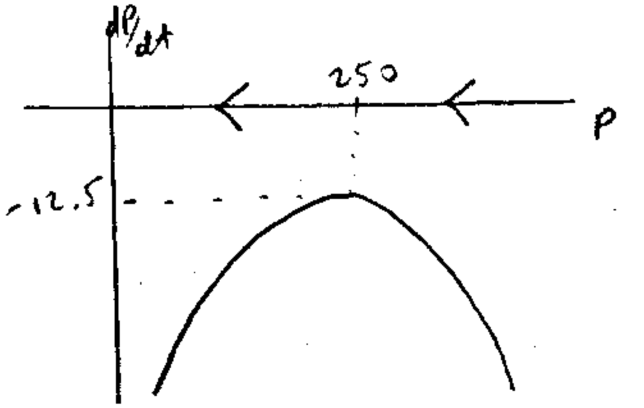
ii) Fish caught at a rate of 75/year :

$$\frac{dP}{dt} = .001\,(500-P)P - 75$$

$$\frac{dP}{dt} = 0 \Rightarrow -P^2 + 500P - 75000 = 0 \Rightarrow$$

$$\Rightarrow \quad P = (-500 \pm i224)/_{-2} = 500 \pm 224i$$

There are no constant solutions for dP/dt



The fish are going to be depleted.