Jan, 2000

Hi, I'm just providing answers for problems 1–4 on page 44 of the SMB, as we took the problems from 41 and 42 off of the required list (i.e. that topic – systems of differential equations – will not be on the exam, indeed many sections did not cover it at all).

p. 44 #1 $\quad y' = xy \quad$ with $y(0) = 1$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

so $xy = \quad a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots$

and $y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$

I shifted these around to keep the same powers of x lined up

so... $a_0 = 1$ from $y(0) = 1$

$a_1 = 0$

$2a_2 = a_0 = 1 \quad$ so $a_2 = \frac{1}{2}$

$3a_3 = a_1 = 0 \quad$ so $a_3 = 0$

$4a_4 = a_2 = \frac{1}{2} \quad$ so $a_4 = \frac{1}{4 \cdot 2}$

$5a_5 = a_3 = 0 \quad$ so $a_5 = 0$

so the solution is

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + \cdots \\ &= 1 + 0 \cdot x + \tfrac{1}{2} x^2 + 0 \cdot x^3 + \tfrac{1}{4 \cdot 2} x^4 + 0 \cdot x^5 + \cdots \\ &= 1 + \frac{x^2}{2} + \frac{x^4}{4 \cdot 2} + \cdots \end{aligned}$$

This one's tough to spot, actually, working backwards, you can solve $y' = xy$ to get $y = Ce^{x^2/2}$, so here $y(x) = e^{x^2/2}$

<u>Then check:</u> $e^x = 1 + x + \frac{x^2}{2} + \cdots$ so $e^{x^2/2} = 1 + \left(\frac{x^2}{2}\right) + \frac{\left(\frac{x^2}{2}\right)^2}{2} + \cdots$

is the same power series

<u>Note!</u> We will not be testing your "recognize the power series" skills on the final, so don't worry if you have trouble with this part of the question!

#2 $\quad y' = x - y \quad$ or $\quad y' + y - x = 0$

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\ y' &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots \\ -x &= \quad -x \end{aligned}$$

2) continued. Now $y(0)=0 \Rightarrow a_0=0$

so $y+y'-x$ tells us that $a_0+a_1=0 \Rightarrow a_1=0$

Then $(a_1+2a_2-1)x=0$ or $0+2a_2-1=0 \Rightarrow a_2=\frac{1}{2}$

$3a_3+a_2=0$, or $3a_3+\frac{1}{2}=0 \Rightarrow a_3=-\frac{1}{3\cdot 2}$

$4a_4+a_3=0$ so $a_4=\frac{1}{4!}$

$5a_5+a_4=0$ so $a_5=\frac{-1}{5!}$

so $y(x)=0+0x+\frac{1}{2}x^2+\left(\frac{-1}{3!}\right)x^3+\frac{1}{4!}x^4+\left(\frac{-1}{5!}\right)x^5+\cdots$

$$=\frac{x^2}{2}-\frac{x^3}{3!}+\frac{x^4}{4!}-\frac{x^5}{5!}+\cdots$$

This looks like e^{-x} minus the first two terms

or $e^{-x}=1-x+\frac{x^2}{2}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots$

so $y(x)$ looks like $e^{-x}+x-1$

(now check $y'=-e^{-x}+1$, so $y'=x-y=x-(e^{-x}+x-1)$
$=-e^{-x}+1$ ✓

so this is in fact the solution)

(3) $y'=x\sin x$, $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$

so if $y=a_0+a_1x+\cdots$

Then $y'=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+\cdots$

and $x\sin x = x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$

$$= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots$$

So $y(0)=1 \Rightarrow a_0=1$

from $y'=x\sin x$ we get

$a_1=0$, $2a_2=0 \Rightarrow a_2=0$

$3a_3=1 \Rightarrow a_3=\frac{1}{3}$

$4a_4=0 \Rightarrow a_4=0$

$5a_5=-\frac{1}{3!} \Rightarrow a_5=-\frac{1}{5\cdot 3!}$

so $y(x)=1+0\cdot x+0\cdot x^2+\frac{1}{3}x^3+0\cdot x^4-\frac{1}{30}x^5+\cdots$

(3) continued or $y(x) = 1 + \frac{1}{3}x^3 - \frac{1}{5!}x^5 + \cdots$

here separation of variables yields the solution
$$y(x) = \sin x - x\cos x + 1$$
and you can check that this has a power series that starts the same way

(4) $xy'' + y' + xy = 0$

if $y = a_0 + a_1x + a_2x^2 + \cdots$

then
$$xy = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + a_4x^5 + a_5x^6 + \cdots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$
$$xy'' = 2a_2x + 6a_3x^2 + 12a_4x^3 + 20a_5x^4 + 30a_6x^5 + \cdots$$

so $y(0) = 1 \Rightarrow a_0 = 1$ $\quad y'(0) = 0 \Rightarrow a_1 = 0$

next $a_0 + 2a_2 + 2a_2 = 0$ or $1 + 4a_2 = 0 \Rightarrow a_2 = -\frac{1}{4}$

$a_1 + 3a_3 + 6a_3 = 0$ or $0 + 9a_3 = 0 \Rightarrow a_3 = 0$

$a_2 + 4a_4 + 12a_4 = 0$ or $-\frac{1}{4} + 16a_4 = 0 \Rightarrow a_4 = \frac{1}{64}$

$a_3 + 5a_5 + 20a_5 = 0$ or $0 + 25a_5 = 0 \Rightarrow a_5 = 0$

so $y(x) = 1 + 0\cdot x - \frac{1}{4}x^2 + 0x^3 + \frac{1}{64}x^4 + \cdots$
$$= 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots$$

(which is the Bessel function, $J_0(x)$, not that you're expected to recognize this!)