11.8

(6) a)

$$\sum_{k=0}^{\infty}\frac{(-1)^k(x-5)^k}{3^k}=\sum_{k=0}^{\infty}\frac{[(-1)(x-5)]^k}{3^k}=\sum_{k=0}^{\infty}\frac{(5-x)^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{5-x}{3}\right)^k$$

This is a geometric that will converge when $\left|\frac{5-x}{3}\right|<1$

$\frac{5-x}{3}<1$ and $\frac{5-x}{3}>-1$ Therefore, $\chi>2$ and $\chi<8$.

At $\chi=2$: $\sum_{k=0}^{\infty}\frac{(-1)^k(x-5)^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{5-2}{3}\right)^k=\sum_{k=0}^{\infty}1^k$ This Diverges

At $\chi=8$: $\sum_{k=0}^{\infty}\frac{(-1)^k(x-5)^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{5-8}{3}\right)^k=\sum_{k=0}^{\infty}(-1)^k$ This also Diverges

Thus, the domain of f is $2<\chi<8$.

(b)

$$f(3)=\sum_{k=0}^{\infty}\frac{(-1)^k(3-5)^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{5-3}{3}\right)^k=\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k=\frac{1}{1-\frac{2}{3}}=3$$

$$f(6)=\sum_{k=0}^{\infty}\frac{(-1)^k(6-5)^k}{3^k}=\sum_{k=0}^{\infty}\left(\frac{5-6}{3}\right)^k=\sum_{k=0}^{\infty}\left(\frac{-1}{3}\right)^k=\frac{1}{1+\frac{1}{3}}=\frac{3}{4}$$

(22) Geometric Series

$\sum_{k=0}^{\infty}\frac{(x-3)^k}{2^k}=\sum_{k=0}^{\infty}\left(\frac{x-3}{2}\right)^k$ This geometric series converges when $\left|\frac{x-3}{2}\right|<1$.

$\frac{x-3}{2}<1$ and $\frac{x-3}{2}>-1$

$\chi<5$ and $\chi>1$

At $c=1$: $\sum_{k=0}^{\infty}\frac{(1-3)^k}{2^k}=\sum_{k=0}^{\infty}\left(\frac{-2}{2}\right)^k=\sum_{k=0}^{\infty}(-1)^k$ This diverges.

At $c=5$: $\sum_{k=0}^{\infty}\frac{(5-3)^k}{2^k}=\sum_{k=0}^{\infty}\left(\frac{2}{2}\right)^k=\sum_{k=0}^{\infty}(1)^k$ This diverges.

OR by Ratio Test

$$p = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x-3}{2}\right| \text{ which must be} < 1$$

$$\frac{x-3}{2} < 1 \text{ and } \frac{x-3}{2} > -1$$

THEREFORE: I.O.C. = (1,5) and R.O.C. = 2

(28) by Ratio Test

$$p = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{\ln(k+1)(x-3)^{k+1}}{k+1} \bullet \frac{k}{(\ln k)(x-3)^k}\right| = \lim_{k\to\infty}\left|\frac{k}{k+1} \bullet \frac{\ln(k+1)}{\ln k} \bullet (x-3)\right| = |x-3| < 1$$

$x-3<1$ and $x-3>-1$ $\qquad 2<x<4$

At $\chi = 2$: $\sum_{k=0}^{\infty}\frac{(\ln k)(2-3)^k}{k} = \sum_{k=0}^{\infty}\frac{(\ln k)(-1)^k}{k}$ Use Alternating Series Test to prove convergence

At $\chi = 4$: $\sum_{k=0}^{\infty}\frac{(\ln k)(4-3)^k}{k} = \sum_{k=0}^{\infty}\frac{(\ln k)(1)^k}{k}$ Does not converge

THEREFORE: I.O.C. = [2,4) and R.O.C. = 1

11.5

(1) a)

$f(x) = e^{-x}$ $\qquad f^{I}(x) = -e^{-x}$ $\qquad f^{II}(x) = e^{-x}$

$f(0) = 1$ $\qquad f^{I}(0) = -1$ $\qquad f^{II}(0) = 1$

$f(x) \approx 1 + (-1)(x) + \frac{x^2}{2} = 1 - x + \frac{x^2}{2}$ *Quadratic Approximation*

$f(x) \approx 1 - x$ *Linear Approximation*

(b)

$f(x) = \cos x$ $\qquad f^{I}(x) = -\sin x$ $\qquad f^{II}(x) = -\cos x$

$f(0) = 1$ $\qquad f^{I}(0) = 0$ $\qquad f^{II}(0) = -1$

$f(x) \approx 1 - \frac{x^2}{2}$ *Quadratic Approximation*

$f(x) \approx 1$ *Linear Approximation*

(c)

f(x) = sin x f^{I}(x) = cos x f^{II}(x) = -sin x
f(π/2) = 1 f^{I}(π/2) = 0 f^{II}(π/2) = -1

$f(x) \approx 1 - \frac{(x-\frac{\pi}{2})^2}{2}$ *Quadratic Approximation*

$f(x) \approx 1$ *Linear Approximation*

(d)

$f(x) = \sqrt{x}$ $f^{I}(x) = \frac{1}{2\sqrt{x}}$ $f^{II}(x) = -\frac{1}{4x^{3/2}}$

f(1) = 1 f^{I}(1) = 1/2 f^{II}(1) = -1/4

$f(x) \approx 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8}$ *Quadratic Approximation*

$f(x) \approx 1 + \frac{(x-1)}{2}$ *Linear Approximation*

(2) a)

$f(x) \approx 1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{8}$ (See 1d above)

(b)

$f(1.1) \approx 1 + \frac{1.1-1}{2} - \frac{(1.1-1)^2}{8} = 1.04875$

$\sqrt{1.1} = 1.048808...$

(7)

$f(x) = e^{-x}$ $f^{I}(x) = -e^{-x}$ $f^{II}(x) = e^{-x}$ $f^{III}(x) = -e^{-x}$ $f^{IV}(x) = e^{-x}$

f(0) = 1 f^{I}(0) = -1 f^{II}(0) = 1 f^{III}(0) = -1 f^{IV}(0) = 1

$f_0(x) = 1$ $f_1(x) = 1 - x$ $f_2(x) = 1 - x + \frac{x^2}{2}$ $f_3(x) \doteq 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$ $f_4(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$

$f_k(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!}$

(8)

$f(x) = e^{ax}$ $\quad f^{I}(x) = ae^{ax}$ $\quad f^{II}(x) = a^2e^{ax}$ $\quad f^{III}(x) = a^3e^{ax}$ $\quad f^{IV}(x) = a^4e^{ax}$

$f(0) = 1$ $\quad f^{I}(0) = a$ $\quad f^{II}(0) = a^2$ $\quad f^{III}(0) = a^3$ $\quad f^{IV}(0) = a^4$

$f_0(x) = 1 \quad f_1(x) = 1 + ax \quad f_2(x) = 1 + ax + \frac{a^2x^2}{2}$

$f_3(x) = 1 + ax + \frac{a^2x^2}{2} - \frac{a^3x^3}{6} \quad f_4(x) = 1 + ax + \frac{a^2x^2}{2} + \frac{a^3x^3}{6} + \frac{a^4x^4}{24}$

$f_k(x) = \sum_{k=0}^{\infty} \frac{a^k x^k}{k!}$

(12)

$f(x) = \frac{1}{1+x}$ $\quad f^{I}(x) = -\frac{1}{(1+x)^2}$ $\quad f^{II}(x) = \frac{2}{(1+x)^3}$ $\quad f^{III}(x) = -\frac{6}{(1+x)^4}$ $\quad f^{IV}(x) = \frac{24}{(1+x)^5}$

$f(0) = 1$ $\quad f^{I}(0) = -1$ $\quad f^{II}(0) = 2$ $\quad f^{III}(0) = 6$ $\quad f^{IV}(0) = 24$

$f_0(x) = 1 \quad f_1(x) = 1 - x \quad f_2(x) = 1 - x + x^2 \quad f_3(x) = 1 - x + x^2 - x^3 \quad f_4(x) = 1 - x + x^2 - x^3 + \frac{x^4}{24}$

$f_k(x) = \sum_{k=0}^{\infty} (-1)^k x^k$

(17) a)

$f(x) = 1 + 2x - x^2 + x^3$ $\quad f^{I}(x) = 2-2x+3x^2$ $\quad f^{II}(x) = -2+6x$ $\quad f^{III}(x) = 6$

$f(0) = 1$ $\quad f^{I}(0) = 2$ $\quad f^{II}(0) = -2$ $\quad f^{III}(0) = 6$

$f(x) = 1 + 2x - 2x^2/2 + 6x^3$

(b)

$f^{(k)}(0) = k!c_k$ for $k \le n$; $f^{(k)}(0) = 0$ for $k>n$. Therefore, the Maclaurin series for f(x) is f(x).

(24)

$f(x) = \cos x$ $\quad f^{I}(x) = -\sin x$ $\quad f^{II}(x) = -\cos x$ $\quad f^{III}(x) = \sin x$ $\quad f^{IV}(x) = \cos x$

$f(\frac{\pi}{2}) = 0$ $\quad f^{I}(\frac{\pi}{2}) = -1$ $\quad f^{II}(\frac{\pi}{2}) = 0$ $\quad f^{III}(\frac{\pi}{2}) = 1$ $\quad f^{IV}(0) = 0$

$f_0(x) = 0$ $\quad f_1(x) = -(x-\frac{\pi}{2})$ $\quad f_2(x) = -(x-\frac{\pi}{2})$

$f_3(x) = -(x-\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{6}$ $\quad f_4(x) = -(x-\frac{\pi}{2}) + \frac{(x-\frac{\pi}{2})^3}{6}$

$$f_k(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(x-\pi/2)^{2k+1}}{(2k+1)!}$$

(26)

$f(x) = \ln x$ $\quad f^{I}(x) = \frac{1}{x}$ $\quad f^{II}(x) = -\frac{1}{x^2}$ $\quad f^{III}(x) = \frac{2}{x^3}$ $\quad f^{IV}(x) = -\frac{6}{x^4}$

$f(e) = 1$ $\quad f^{I}(e) = \frac{1}{e}$ $\quad f^{II}(e) = -\frac{1}{e^2}$ $\quad f^{III}(e) = \frac{2}{e^3}$ $\quad f^{IV}(e) = -\frac{6}{e^4}$

$f_0(x) = 1$ $\quad f_1(x) = 1 + \frac{1}{e}(x-e)$ $\quad f_2(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2$

$f_3(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{2}{6e^3}(x-e)^3$ $\quad f_4(x) = 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{2}{6e^3}(x-e)^3 - \frac{6}{24e^4}(x-e)^4$

$$f_k(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(k-1)!}{k!e^k}(x-e)^k = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{ke^k}(x-e)^k$$

(33)

- $f(0) = 1$; therefore, positive at x=0.
- 2nd term is negative; therefore First Derivative is negative at x=0 =>graph decreasing.
- 3rd term is positive; therefore Second Derivative is positive => concave up.

Choice IV is both decreasing and concave up.

11.10

(1)

$$\frac{1}{1-x}=\sum_{k=0}^{\infty}x^k$$

Therefore:

(a)

$\frac{1}{1+x}=\sum_{k=0}^{\infty}(-x)^k=1-x+x^2-x^3+\cdots$ when $|-x|<1$. ROC = 1.

(b)

$\frac{1}{1-x^2}=\sum_{k=0}^{\infty}(x^2)^k=1+x^2+x^4+\cdots$ when $(x^2)<1$. ROC = 1.

(c)

$\frac{1}{1-2x}=\sum_{k=0}^{\infty}(2x)^k=1+2x+4x^2+8x^3+\cdots$ when $|2x|<1$. ROC = $\frac{1}{2}$.

(d)

$\frac{1}{2-x}=\frac{1/2}{1-(x/2)}=\frac{1}{2}+\frac{1}{2^2}x+\frac{1}{2^3}x^2+\frac{1}{2^4}x^3+\cdots$ when $\left|\frac{x}{2}\right|<1$. ROC = 2.

(6)

(a)

$$\cos x = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots$$

$$\cos 2x = 1-\frac{1}{2!}(2x)^2+\frac{1}{4!}(2x)^4-\frac{1}{6!}(2x)^6+\cdots=1-\frac{2^2}{2!}x^2+\frac{2^4}{4!}x^4-\frac{2^6}{6!}x^6+\cdots=\sum_{k=0}^{\infty}\frac{(2x)^{2k}}{(2k)!}$$

$$\lim_{k\to\infty}\left[\frac{(2x)^{2(k+1)}}{(2(k+1))!}\bullet\frac{(2k)!}{(2x)^{2k}}\right]=\lim_{k\to\infty}\frac{(2x)^2}{(2k+2)(2k+1)}=0 \text{ for all values of x.}$$

ROC = ∞

(b)

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$x^2e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$

$\lim_{k\to\infty}\left[\frac{x^{(k+1)+2}}{(k+1)!} \bullet \frac{k!}{x^{k+2}}\right] = \lim_{k\to\infty} \frac{x}{(k+1)} = 0$ for all values of x.

ROC = ∞

(c)

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$xe^{-x} = x + x(-x) + \frac{x(-x)^2}{2!} + \frac{x(-x)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (x^{k+1})}{k!}$

$\lim_{k\to\infty}\left|\frac{x^{(k+1)+1}}{(k+1)!} \bullet \frac{k!}{x^{k+1}}\right| = \lim_{k\to\infty} \frac{x}{(k+1)} = 0$ for all values of x.

ROC = ∞

(d)

$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$

$\sin (x^2) = x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \cdots = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k (x^{4k+2})}{(2k+1)!}$

$\lim_{k\to\infty}\left|\frac{x^{4(k+1)+2}}{(2(k+1)+1)!} \bullet \frac{(2k+1)!}{x^{4k+2}}\right| = \lim_{k\to\infty} \frac{x^{4k+6}(2k+1)!}{(2k+3)!(x^{4k+1})} = \lim_{k\to\infty} \frac{x^5}{(2k+3)(2k+2)} = 0$ for all values of x.

ROC = ∞

(21)

(a)

$\frac{d}{dx}\cos x = \frac{d}{dx}\left[1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots = -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = -\sin x$

(b)

$$\frac{d}{dx}\ln(1+x)=\frac{d}{dx}\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right]=1-x+x^2-x^3+\cdots=\sum_{k=0}^{\infty}(-x)^k=\frac{1}{1+x}$$

(33)(a)

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4\cdots$$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right]=-\frac{1}{(1-x)^2}$$

$$\frac{d}{dx}\left[1+x+x^2+x^3+x^4+\cdots\right]=1+2x+3x^2+4x^3+\cdots$$

$$\frac{x}{(1-x)^2}=x\frac{d}{dx}\left[\frac{1}{1-x}\right]=x\bullet\frac{d}{dx}\left[1+x+x^2+x^3+x^4+\cdots\right]=x+2x^2+3x^3+4x^4+\cdots=\sum_{k=1}^{\infty}kx^k$$

(34)(a)

$$\sum_{k=1}^{\infty}k\left(\frac{1}{3}\right)^k=\frac{1/3}{(1-1/3)^2}=\frac{1/3}{(2/3)^2}=\frac{1/3}{4/9}=\frac{3}{4}$$