9.8. #2

(a) $\int_0^1 x^{-p}\,dx$ improper for $\underline{p>0}$,

because $x^{-p} \to \infty$ as $x \to 0$ for $p>0$

$x^{-p} = 1$ for $p=0$

$x^{-p} \to 0$ as $x \to 0$ for $p<0$.

(b) $\int_1^2 \frac{dx}{x-p}$ improper for $\underline{1 \leq p \leq 2}$,

because $\frac{1}{x-p} \to \pm\infty$ as $x \to p$, and since

$1 \leq x \leq 2$ for ~~this~~ definite integral,

it follows that $1 \leq p \leq 2$ for the integral to be improper.

(c) $\int_0^1 \frac{dx}{e^{px}} = \int_0^1 e^{-px}\,dx$ - continuous for all values of p,

so the integral is $\underline{\text{not}}$ improper for $\underline{\text{any } p}$.

9.8. #4

$$\int_{-1}^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_{-1}^{\infty} \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln\left(x^2+1\right)\Big|_{-1}^{\infty} = \underbrace{\frac{1}{2}\lim_{l\to\infty} \ln\left(l^2+1\right)}_{\text{diverges}} - \frac{1}{2}\ln\left[(-1)^2+1\right]$$

$\Rightarrow$ Integral $\underline{\text{does not}}$ converge

9.8. #6.

$$\int_0^{\infty} x e^{-x^2}\,dx = -\frac{1}{2}\int_0^{\infty}(-2x)e^{-x^2}\,dx = -\frac{1}{2}\, e^{-x^2}\Big|_0^{\infty} = -\frac{1}{2}\lim_{l\to\infty} e^{-l^2} + \frac{1}{2}e^{-0^2}$$

$$= -\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \underline{\underline{\frac{1}{2}}}$$

9.8. #10.

$$\int_{-\infty}^{2} \frac{dx}{x^2+4} = \frac{1}{4}\int_{-\infty}^{2} \frac{dx}{1+\left(\frac{x}{2}\right)^2} = \frac{1}{4}\int_{-\infty}^{1} \frac{2\,du}{1+u^2}$$

$u = \frac{x}{2}$

$dx = 2\,du$

$$= \frac{1}{2}\int_{-\infty}^{1} \frac{du}{1+u^2} = \frac{1}{2}\arctan u\Big|_{-\infty}^{1} = \frac{1}{2}\arctan 1 - \frac{1}{2}\lim_{l\to-\infty} \arctan l =$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8} + \frac{\pi}{4} = \underline{\underline{\frac{3\pi}{8}}}$$

9.8. #16

$$\int_{-\infty}^{\infty} \frac{e^{-t}}{1+e^{-2t}}\, dt =$$

$u = e^{-t}$

$\frac{du}{dt} = -e^{-t} \Rightarrow dt = \frac{du}{-e^{-t}} = \frac{du}{-u}$

$$= \int_{\infty}^{0} \frac{\cancel{u}}{1+u^2}\left(\frac{du}{-\cancel{u}}\right) = \int_{0}^{\infty} \frac{du}{1+u^2} = \arctan u \,\Big|_0^{\infty} =$$

$$= \lim_{l \to \infty} \arctan u - \arctan 0 = \frac{\pi}{2} - 0 = \underline{\underline{\frac{\pi}{2}}}$$

9.8. #27.

$$\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\, dx = \int_{-1}^{8} x^{-\frac{1}{3}}\, dx$$

This is a case as in 9.8. #2(a), with $p = \frac{1}{3}$. Since $p > 0$, hence the integral is improper and so we must split it into two:

$$= \int_{-1}^{0} x^{-\frac{1}{3}}\, dx + \int_{0}^{8} x^{-\frac{1}{3}}\, dx =$$

$$= \frac{3}{2} x^{\frac{2}{3}} \Big|_{-1}^{0} + \frac{3}{2} x^{\frac{2}{3}} \Big|_{0}^{8} = \underbrace{\cancel{\frac{3}{2} 0^{\frac{2}{3}}}}_{(1)} - \underbrace{\frac{3}{2}(-1)^{\frac{2}{3}}}_{(A)} + \underbrace{\frac{3}{2} 8^{\frac{2}{3}}}_{(B)} - \underbrace{\cancel{\frac{3}{2} 0^{\frac{2}{3}}}}_{(2)} =$$

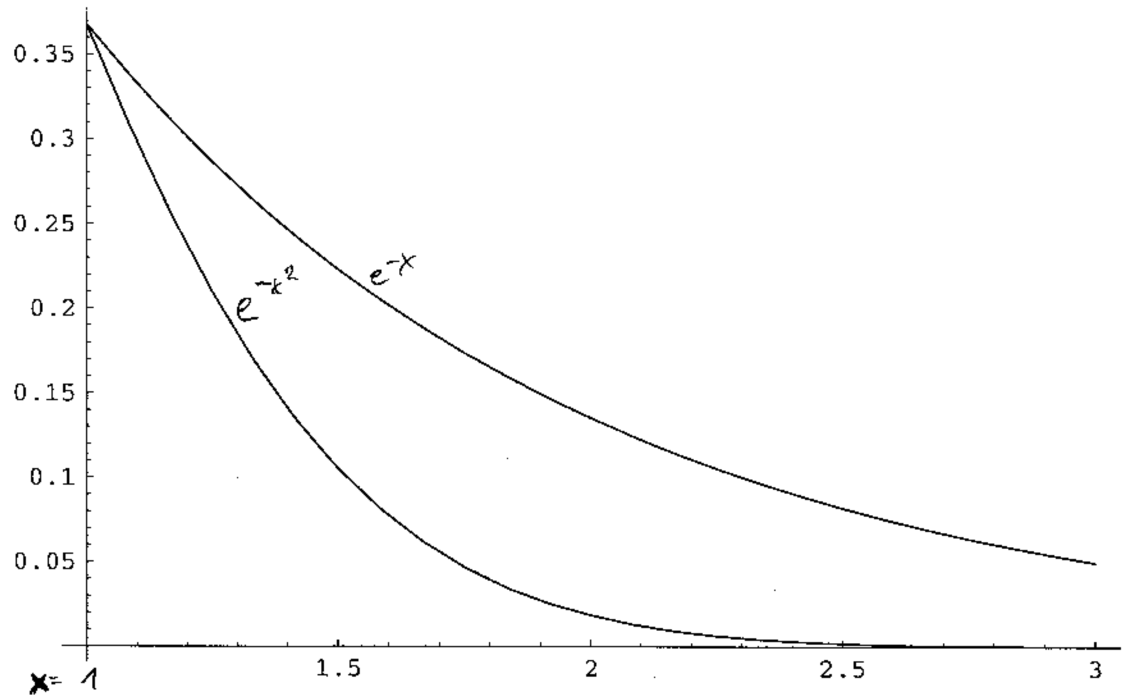
$$= -\frac{3}{2} \cdot 1 + \frac{3}{2} \cdot 4 = -\frac{3}{2} + 6 = \underline{\underline{\frac{9}{2}}}$$

Notice that $\int_{-1}^{8} x^{-\frac{1}{3}}\, dx = \frac{3}{2} x^{\frac{2}{3}} \Big|_{-1}^{8} = \underbrace{\frac{3}{2} 8^{\frac{2}{3}}}_{(B)} - \underbrace{\frac{3}{2}(-1)^{\frac{2}{3}}}_{(A)} = \frac{9}{2}$

gives the same result in <u>this</u> case, but in general does not ensure that (1) and (2) from the first expression cancel each other out — it does not check for convergence. The second method is not valid in general — applicable only when you are sure that elements corresponding to (1) and (2) above will cancel.

9.8. #47

(a)



So, graphically, $e^{-x^2} \leqslant e^{-x}$ for $x \geqslant 1$

(b) Algebraically $e^{-x^2} \leq e^{-x}$

$-x^2 \leqslant -x$ Taking ln of both sides

$x^2 \geqslant x \Rightarrow x \geqslant 1$ or $x \leqslant 0$

(b) $\int_1^\infty e^{-x}dx = -e^{-x}\Big|_1^\infty = -\lim_{l\to\infty} e^{-l} - \left(-e^{-1}\right) = -0 + e^{-1} = \frac{1}{e}$

(c) Since $e^{-x^2} \leqslant e^{-x}$ on $[1, +\infty)$,

hence

$$\int_1^\infty e^{-x^2}dx \leqslant \int_1^\infty e^{-x}dx = \frac{1}{e}$$

$$\int_1^\infty e^{-x^2}dx < \frac{1}{e}$$

8.2.#1

$$\pi \int_{-1}^{3} \left(\sqrt{3-x}\right)^2 dx = \pi \int_{-1}^{3} (3-x)\,dx = \pi \left(3x - \frac{x^2}{2}\right)\Bigg|_{-1}^{3} =$$

$$= \pi \left[\left(3\cdot 3 - \frac{3^2}{2}\right) - \left(3\cdot(-1) - \frac{(-1)^2}{2}\right)\right] = \pi \left[4.5 - (-3.5)\right] = \underline{\underline{8\pi}}$$

8.2.#2

The right limit of integration is when $x = 2 - x^2 \Rightarrow \underline{x=1}$

$$\pi \int_0^1 \left[(2-x^2)^2 - (x)^2\right] dx = \pi \int_0^1 (4 - 4x^2 + x^4 - x^2)\,dx = \pi \int_0^1 (x^4 - 5x^2 + 4)\,dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{5}{3}x^3 + 4x\right)\Bigg|_0^1 = \pi \left(\frac{1^5}{5} - \frac{5}{3}1^3 + 4\cdot 1\right) - \pi\cdot 0 =$$

$$= \pi \left(\frac{1}{5} - \frac{5}{3} + 4\right) = \underline{\underline{\frac{38}{15}\pi}}$$

8.2.#8

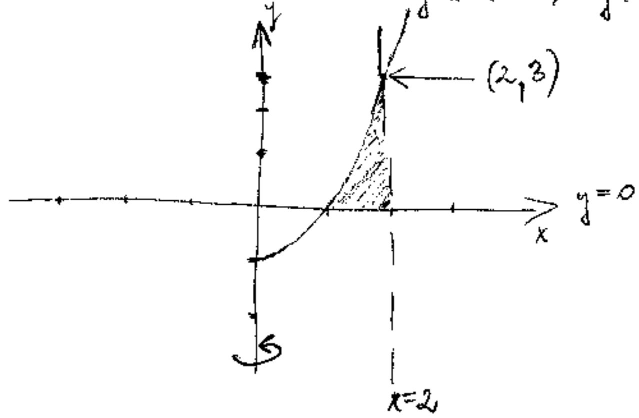
Again, the upper and lower limits of integration are where $x^2 = x^3$
$\Rightarrow x = 0, 1$

Since $x^2 > x^3$ on $(0,1)$, therefore we shall be integrating $\pi \int \left[(x^2)^2 - (x^3)^2\right]$ over this interval.

$$\pi \int_0^1 (x^4 - x^6)\,dx = \pi \left(\frac{x^5}{5} - \frac{x^7}{7}\right)\Bigg|_0^1 = \pi \left(\frac{1^5}{5} - \frac{1^7}{7}\right) - \pi\cdot 0 = \pi\left(\frac{1}{5} - \frac{1}{7}\right) = \underline{\underline{\frac{2}{35}\pi}}$$

8.2.#18

$y = x^2 - 1 \Rightarrow y + 1 = x^2 \Rightarrow x = \sqrt{y+1}$



$\underline{y=0}$ is the lower limit of $\int$

$y = 3$ is the upper limit,

because $\begin{cases} x = 2 \\ y = x^2 - 1 \end{cases} \Rightarrow \underline{y = 3}$

$$\pi\int_0^3 \left((2)^2-(\sqrt{y-1})^2\right)dy = \pi\int_0^3 \left(4-(y-1)\right)dy = \pi\int_0^3 (5-y)\,dy = \pi\left(5y-\frac{y^2}{2}\right)\Big|_0^3 =$$

$$= \pi\left(5\cdot 3-\frac{3^2}{2}\right) - \pi\cdot 0 = \pi\left(15-\frac{9}{2}\right) = \underline{\underline{\frac{21}{2}\pi}}$$

8.2. #35

(a) We want $\int_{-1}^{1} A(x)\,dx$, where $A(x)$ is the area of the cross section of the solid at x.

Here $A(x) = \frac{1}{2}\pi y^2 = \frac{1}{2}\pi(1-x^2)$ - since $y^2 = 1-x^2$, $1=x^2+y^2$

So
$$\int_{-1}^{1}\frac{1}{2}\pi(1-x^2)\,dx = \frac{1}{2}\pi\int_{-1}^{1}(1-x^2)\,dx = \frac{1}{2}\pi\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} =$$
$$= \frac{1}{2}\pi\left(1-\frac{1^3}{3}\right) - \frac{1}{2}\pi\left(-1-\frac{(-1)^3}{3}\right) = \frac{1}{2}\pi\cdot\frac{2}{3} - \frac{1}{2}\pi\left(-\frac{2}{3}\right) = \underline{\underline{\frac{2}{3}\pi}}$$

which makes sense because the solid is just a half sphere of $r=1$.

(b) $A(x) = (2y)^2 = 4(1-x)^2$

$$4\int_{-1}^{1}(1-x^2)\,dy = 4\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = 4\left[1-\frac{1}{3}-(-1)+\frac{(-1)^3}{3}\right] = 4\left[1-\frac{1}{3}+1-\frac{1}{3}\right] = 4\cdot\frac{4}{3} = \underline{\underline{\frac{16}{3}}}$$

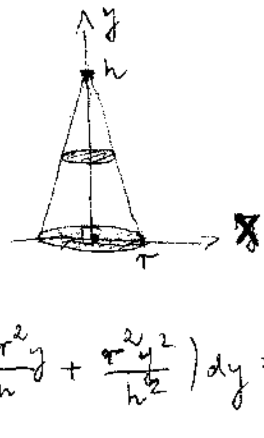
(c) $A(x) = \frac{(2y)^2\sqrt{3}}{4}$, since cross section is an equilateral triangle, the formula for whose area is $\frac{a^2\sqrt{3}}{4}$. Here $a=2y$.

$$= \frac{4(1-x^2)\sqrt{3}}{4} = \sqrt{3}\,(1-x^2)$$

$$\sqrt{3}\int_{-1}^{1}(1-x^2)\,dy = \sqrt{3}\left(x-\frac{x^3}{3}\right)\Big|_{-1}^{1} = \sqrt{3}\cdot\frac{4}{3} = \underline{\underline{\frac{4\sqrt{3}}{3}}}$$

8.2. #36

$$V = \int_0^h \pi r(y)^2\,dy$$



From the picture you can see that $r(y)$ is a straight line passing through $(r,0)$ & $(0,h)$

$$\Rightarrow r(y) = -\frac{r}{h}y + r$$

$$\pi\int_0^h\left(r-\frac{r}{h}y\right)^2 dy = \pi\int_0^h\left(r^2-\frac{2r^2y}{h}+\frac{r^2y^2}{h^2}\right)dy = \pi\left[r^2y-\frac{2r^2y^2}{2h}+\frac{r^2y^3}{3h^2}\right]_0^h =$$

$$= \pi\left[\left(r^2h - \frac{r^2h^2}{\cancel{h}} + \frac{r^2h^2}{3\cancel{h^2}}\right) - 0\right] = \pi\left(\cancel{r^2}h - \cancel{r^2}h + \frac{r^2h}{3}\right) = \frac{\pi r^2 h}{3}$$

8.2. #45

We want $\int_{-r}^{r} A(y)\,dy$.

The cross section, whose area A represents, is a triangle with base x, height $x\tan\theta$, so $A(y) = \frac{1}{2}x \cdot x\tan\theta = \frac{x^2\tan\theta}{2}$

As the base of the cylinder is circular, with radius r, we have $x^2+y^2=r^2$

$\Rightarrow \; A(y) = \frac{(r^2-y^2)\tan\theta}{2}$

$$\int_{-r}^{r} A(y)\,dy = \int_{-r}^{r} \frac{(r^2-y^2)}{2}\tan\theta\,dy = \frac{\tan\theta}{2}\left(r^2y - \frac{y^3}{3}\right)\Big|_{-r}^{r} =$$

$$= \frac{\tan\theta}{2}\left[\left(r^2r - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right] = \frac{\tan\theta}{2}\left[r^3 - \frac{r^3}{3} + r^3 - \frac{r^3}{3}\right] =$$

$$= \frac{2}{3}\tan\theta\, r^3$$

8.2. #47

$\frac{1}{8}V = \int_0^r A(z)\,dz$. The cross section is a square, so:

$A(z) = x^2 = r^2 - z^2$, since we have $x^2+z^2=r^2$; the base of the cylinder is a circle in the xz plane.

$$\frac{1}{8}V = \int_0^r (r^2-z^2)\,dz = r^2z - \frac{z^3}{3}\Big|_0^r = r^3 - \frac{r^3}{3} - 0 = \frac{2}{3}r^3$$

$$V = \frac{16}{3}r^3$$

8.4. #6

$$y = \frac{x^6+8}{16x^2} \Rightarrow \frac{dy}{dx} = \frac{x^3}{4} - \frac{1}{x^3} \qquad \left(\frac{dy}{dx}\right)^2 = \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6}$$

For arc:

$$\int_2^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_2^3 \sqrt{1+\frac{x^6}{16}-\frac{1}{2}+\frac{1}{x^6}}\,dx = \int_2^3 \sqrt{\frac{x^6}{16}+\frac{1}{2}+\frac{1}{x^6}}\,dx = \int_2^3 \sqrt{\left(\frac{x^3}{4}+\frac{1}{x^3}\right)^2}\,dx = \int_2^3 \left(\frac{x^3}{4}+\frac{1}{x^3}\right)dx =$$

$$= \frac{x^4}{16} - \frac{1}{2x^2}\Big|_2^3 = \frac{3^4}{16} - \frac{1}{2\cdot 3^2} - \frac{2^4}{16} + \frac{1}{2\cdot 2^2} = \frac{81}{16} - \frac{1}{18} - 1 + \frac{1}{8} =$$

$$= \frac{9\cdot 81 - 8 - 144 + 18}{144} = \underline{\underline{\frac{595}{144}}}$$

8.4.#7

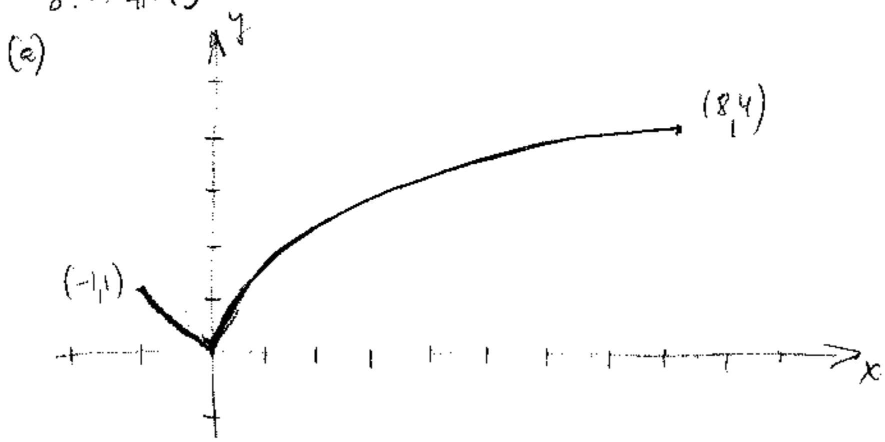
$$y = \frac{e^x + e^{-x}}{2} \Rightarrow \frac{dy}{dx} = \frac{e^x - e^{-x}}{2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 \sqrt{\frac{4 + e^{2x} - 2 + e^{-2x}}{4}}\, dx = \int_0^3 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}\, dx = \int_0^3 \sqrt{\left(\frac{e^x + e^{-x}}{2}\right)^2}\, dx$$

$$= \int_0^3 \frac{e^x + e^{-x}}{2}\, dx = \left(\frac{e^x}{2} - \frac{e^{-x}}{2}\right)\Big|_0^3 = \frac{e^3}{2} - \frac{e^{-3}}{2} - \cancel{\frac{e^0}{2}} + \cancel{\frac{e^{-0}}{2}} = \underline{\underline{\frac{e^3}{2} - \frac{1}{2e^3}}}$$

8.4. #19

(a)



(b) The curve "bends" at 0, so $\frac{dy}{dx}$ does not exist at $x=0$.

(c) Need to split the arc-length integral into two portions:

$$\int_{-1}^{0} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx + \int_0^8 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Here } \frac{dy}{dx} = \frac{2}{3}x^{-1/3}, \text{ so}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(\frac{2}{3}x^{-1/3}\right)^2} = \sqrt{1+\frac{4}{9x^{2/3}}} = \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}$$

continued on next page

(c) continued

when $x>0$ this equals $\dfrac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}$

and when x is negative, you have to be careful about taking square roots! you actually get $-\dfrac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}$

So the arc-length equals

$$\int_{-1}^{0} -\frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx + \int_{0}^{8} \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx,$$ both improper integrals

either integral can be solved using substitution now:

let $u = 9x^{2/3}+4$, then $du = 6x^{-1/3}dx$, so $\frac{1}{6}du = \frac{dx}{x^{1/3}}$

so

$$\int \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \int \frac{1}{6}\frac{\sqrt{u}}{3}\,du = \frac{1}{18}\cdot\frac{2}{3}u^{3/2} = \frac{(9x^{2/3}+4)^{3/2}}{27} + C$$

so $$\int_{-1}^{0} -\frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \lim_{N\to 0^-}\left(\int_{-1}^{N} -\frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx\right) = \lim_{N\to 0^-}\left(-\frac{(9x^{2/3}+4)^{3/2}}{27}\Bigg|_{-1}^{N}\right)$$

$$= -\frac{4^{3/2}}{27} - \left(-\frac{13^{3/2}}{27}\right)$$

and likewise $$\int_{0}^{8} \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \lim_{N\to 0^+}\left(\frac{(9x^{2/3}+4)^{3/2}}{27}\Bigg|_{N}^{8}\right)$$

$$= \frac{(9\cdot 8^{2/3}+4)^{3/2}}{27} - \frac{4^{3/2}}{27}$$

So all together you get –

$$\frac{(9\cdot 8^{2/3}+4)^{3/2}}{27} - \frac{4^{3/2}}{27} + \left(-\frac{4^{3/2}}{27} + \frac{13^{3/2}}{27}\right)$$

$$= \frac{40\sqrt{40}}{27} - \frac{8}{27} + \left(-\frac{8}{27}\right) + \frac{13\sqrt{13}}{27} = \frac{80\sqrt{10}}{27} + \frac{13\sqrt{13}}{27} - \frac{16}{27}$$

yea!