Homework for Math 1b

4a) $2\dfrac{dy}{dx} + y = x - 1$ $\boxed{\text{order is } 1}$

$$y = ce^{-x/2} + x - 3$$
$$\frac{dy}{dx} = -\frac{c}{2}e^{-x/2} + 1$$
$$2\left(-\frac{c}{2}e^{-x^2/2} + 1\right) + ce^{-x/2} + x - 3 = x - 1$$
$$x - 1 = x - 1 \checkmark$$

b) $y'' - y = 0$ $\boxed{\text{order is } 2}$

$$y = c_1e^{t} + c_2e^{-t}$$
$$y' = c_1e^{t} - c_2e^{-t}$$
$$y'' = c_1e^{t} + c_2e^{-t}$$

$$c_1e^{t} + c_2e^{-t} - (c_1e^{t} + c_2e^{-t}) = 0$$
$$0 = 0 \checkmark$$

8. a) Integrating factors:

$$\frac{dy}{dx} - \underbrace{4xy}_{} = \underbrace{0}_{}$$
$$p(x) = -4x \quad q(x) = 0$$

$$\mu = e^{\int p(x)dx}$$
$$\mu = e^{-4\int x\,dx} = e^{-2x^2}$$

$$\frac{d}{dx}\left[e^{-2x^2}y\right] = \mu \cdot 0 = 0$$
$$e^{-2x^2}y = C$$
$$\boxed{y = Ce^{2x^2}}$$

Homework for Math 16

8 a) Separation of Variables

$$\frac{dy}{dx} = 4xy$$

$$\frac{dy}{y} = 4x\,dx$$

$$\ln|y| = 2x^2 + C_1$$

$$|y| = e^{2x^2} e^{C_1}$$

$$y = \pm e^{C_1} e^{2x^2}$$

$$\boxed{y = C e^{2x^2}}$$

8 b) Integrating Factors:

$$\frac{dy}{dt} + y = 0$$

$$p(t) = 1 \qquad q(t) = 0$$

$$\mu = e^{\int p(t)\,dt} = e^{\int 1\,dt} = e^{t}$$

$$\frac{d}{dt}[\mu y] = \mu\, q(t)$$

$$\frac{d}{dt}[e^{t} y] = 0$$

$$e^{t} y = C$$

$$\boxed{y = C e^{-t}}$$

Homework for Math 1b

8 b) Separation of Variables:

$$\frac{dy}{dt} = -y$$

$$\int \frac{dy}{y} = -\int dt$$

$$\ln|y| = -t + C_1$$

$$e^{|y|} = e^{-t} e^{C_1}$$

$$|y| = e^{C_1} e^{-t}$$

$$y = \pm C_1 e^{-t}$$

$$\boxed{y = C e^{-t}}$$

10. $$\frac{dy}{dx} = (1+y^2)\,x^2$$

$$\int \frac{dy}{1+y^2} = \int x^2\,dx$$

$$\tan^{-1}(y) = \frac{x^3}{3} + C$$

$$\tan(\tan^{-1}(y)) = \tan\left(\frac{x^3}{3} + C\right)$$

$$\boxed{y = \tan\left(\frac{x^3}{3} + C\right)}$$

11. $$\frac{\sqrt{1+x^2}}{1+y}\frac{dy}{dx} = -x$$

$$\int \frac{dy}{1+y} = -\frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}\,dx \qquad u = 1+x^2 \quad du = 2x \qquad \int u^{-1/2}\,du = 2u^{1/2} + C$$

$$\ln|1+y| = -\frac{1}{2}\,2(1+x^2)^{1/2} + C$$

$$\ln|1+y| = -(1+x^2)^{1/2} + C$$

$$1+y = e^{-(1+x^2)^{1/2}} e^{C}$$

$$\boxed{y = -1 + Ce^{-(1+x^2)^{1/2}}}$$

Homework for Math 16:

15. $e^{-y}\sin x - y' \cos^2 x = 0$

$$e^{-y}\sin x = y' \cos^2 x$$

$$e^{-y}\sin x = \frac{dy}{dx} \cos^2 x$$

$$\frac{\sin x}{\cos^2 x}\,dx = e^y\,dy$$

$$-1\int -\frac{\sin x}{\cos^2 x}\,dx = \int e^y\,dy$$

let $u = \cos x$ $\quad \int \frac{1}{u^2}\,du = -\frac{1}{u}$

$du = -\sin x$

$$\frac{1}{\cos x} + C = e^y$$

$$y = \ln\left|\frac{1}{\cos x} + C\right|$$

$$\boxed{y = \ln|\sec x + c|}$$

19. $\underbrace{\frac{dy}{dx} + 3y}_{p(x)=3} = \underbrace{e^{-2x}}_{q(x)=e^{-2x}}$ $\quad \mu = e^{\int p(x)\,dx} = e^{3\int dx} = e^{3x}$

$$\frac{d}{dx}[\mu y] = \mu q(x)$$

$$\frac{d}{dx}[e^{3x}y] = e^{3x}e^{-2x}$$

$$\frac{d}{dx}[e^{3x}y] = e^x$$

$$e^{3x}y = e^x + C$$

$$\boxed{y = \frac{e^x + C}{e^{3x}} = e^{-2x} + Ce^{-3x}}$$

20. $\frac{dy}{dx} + \underbrace{2xy}_{p(x) = 2x} = \underbrace{x}_{q(x) = x}$

$$\mu = e^{\int p(x)\,dx} = e^{2\int x\,dx} = e^{x^2}$$

$$\frac{d[\mu y]}{dx} = \mu\, q(x)$$

$$\frac{d[e^{x^2}y]}{dx} = e^{x^2}x$$

$$\int \frac{d[e^{x^2}y]}{dx}\,dx = \frac{1}{2}\int 2x\,e^{x^2}\,dx$$

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C$$

$$\boxed{y = \frac{1}{2} + Ce^{-x^2}}$$

23. $(x^2+1)\frac{dy}{dx} + xy = 0$

$$\frac{dy}{dx} + \underbrace{\frac{x}{x^2+1}}y = 0$$

$p(x) = \frac{x}{x^2+1}$ $\quad q(x) = 0$

$$\mu = e^{\int p(x)\,dx} = e^{\int \frac{x}{x^2+1}\,dx}$$

$$\mu = e^{\frac{1}{2}\int \frac{2x}{x^2+1}\,dx}$$

$$\mu = e^{\frac{1}{2}\ln(x^2+1)}$$

$$\mu = e^{\ln(x^2+1)^{1/2}}$$

$$\mu = (x^2+1)^{1/2}$$

$$\frac{d[\mu y]}{dx} = \mu\, q(x)$$

$$\frac{d[(x^2+1)^{1/2}y]}{dx} = (x^2+1)^{1/2}\cdot 0 = 0$$

$$(x^2+1)^{1/2}y = C$$

$$\boxed{y = \frac{C}{(x^2+1)^{1/2}}}$$

Homework for Math 16:

26)
$$\frac{dy}{dx} = xy$$

Separation:
$$\frac{dy}{y} = x\,dx$$
$$\ln|y| = \frac{x^2}{2} + C$$
$$e^{|y|} = e^{x^2/2} e^{c}$$
$$\boxed{y = c\,e^{x^2/2}}$$

Integrating factors:
$$\frac{dy}{dx} - \underbrace{xy}_{} = 0 \quad q(x) = 0$$
$$\mu = e^{\int p(x)\,dx} = e^{-\int x\,dx} = e^{-x^2/2} \quad p(x) = -x$$
$$\frac{d[\mu y]}{dx} = \mu\, q(x)$$
$$\frac{d\left[e^{-x^2/2} y\right]}{dx} = 0$$
$$e^{-x^2/2} y = c$$
$$\boxed{y = c\,e^{x^2/2}}$$

(a) $y(0) = 1 \Rightarrow 1 = c\,e^{0}$

$C = 1$

$$\boxed{y = e^{x^2/2}}$$

(b) $y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = c\,e^{0}$

$C = \frac{1}{2}$

$$\boxed{y = \frac{1}{2} e^{x^2/2}}$$

41) slope of tangent line at any point (x,y) is xe^y =>

$$\frac{dy}{dx} = x e^y$$

Separation of Variables:

$$e^{-y}\,dy = x\,dx$$
$$\int e^{-y}\,dy = \int x\,dx$$

$$-e^{-y} = \frac{x^2}{2} + C$$

if x intercept at 2, when $y = 0$, $x = 2$.

$$-e^0 = \frac{4}{2} + C$$
$$C = -3$$

$$-e^{-y} = \frac{x^2}{2} - 3$$
$$\boxed{2e^{-y} = x^2 + 6}$$

or $$e^{-y} = \frac{1}{2}x^2 + 3$$

$$-y = \ln\left(\frac{1}{2}x^2 + 3\right)$$
$$y = -\ln\left(\frac{1}{2}x^2 + 3\right)$$

Homework for Math 1b

43. $\frac{dy}{dt}$ = rate in − rate out

Let $y(t)$ = amount of salt (in ounces) after t minutes

rate in: $\left(4\ \frac{oz}{gal}\right)\left(2\ \frac{gal}{min}\right) = 8\ \frac{oz}{min}$

amount of salt water stays constant

rate out: $\left(\frac{y(t)}{50}\ \frac{oz}{gal}\right)\left(2\ \frac{gal}{min}\right) = \frac{y(t)}{25}\ \frac{oz}{min}$

$$\frac{dy}{dt} = \text{rate in} - \text{rate out} = 8 - \frac{y(t)}{25}$$

$$\frac{dy}{dt} + \frac{y(t)}{25} = 8$$

Integrating factors: $p(t) = \frac{1}{25}$ $q(t) = 8$

$$\mu = e^{\int p(t)dt} = e^{\frac{1}{25}\int dt} = e^{\frac{t}{25}}$$

$$\frac{d[\mu y]}{dt} = \mu q(x)$$

$$\frac{d[e^{t/25}y]}{dt} = e^{t/25}\, 8$$

$$e^{t/25}y = 8\int e^{t/25}dt$$

$$e^{t/25}y = 25 \cdot 8\ e^{t/25} + C$$

$$y = 200 + Ce^{-t/25}$$

when $t=0$, $y=25$ => $25 = 200 + Ce^0$ => $C = -175$

$\boxed{y(t) = 200 - 175e^{-1/25 t}}$

b) when $t = 25\text{min}$

$y(25) = 200 - 175e^{-\frac{25}{25}}$

$\boxed{y(25) = 200 - \frac{175}{e} = 135.6\ oz}$

45) The volume of the water in the tank at time t is $500 + (\underset{\text{rate in}}{20} - \underset{\text{rate out}}{10})t = 500 + 10t$,

so it will overflow when $t = 50$ minutes

If $y(t)$ = the number of pounds of particulate matter in the water, then $y(0) = 50$ and

$$\frac{dy}{dt} = \underset{\text{rate in}}{0} - \underset{\text{rate out}}{10\left(\frac{y(t)}{500+10t}\right)} = \frac{-1}{50+t}\, y$$

(of particulate matter)

so $\frac{dy}{dt} + \frac{1}{50+t} y = 0$, and the integrating factor is $e^{\int \frac{dt}{50+t}} = 50+t$

so $\frac{d}{dt}\left[(50+t)y\right] = 0$, and $(50+t)\,y = C$

when $t=0$ get $50\,y(0) = 2500 = C$,

so $y(t) = \dfrac{2500}{50+t}$

Then at $t=50$, when tank over flows there is $y(50) = \dfrac{2500}{50+50} = 25$ lb. of particulate matter in the tank.