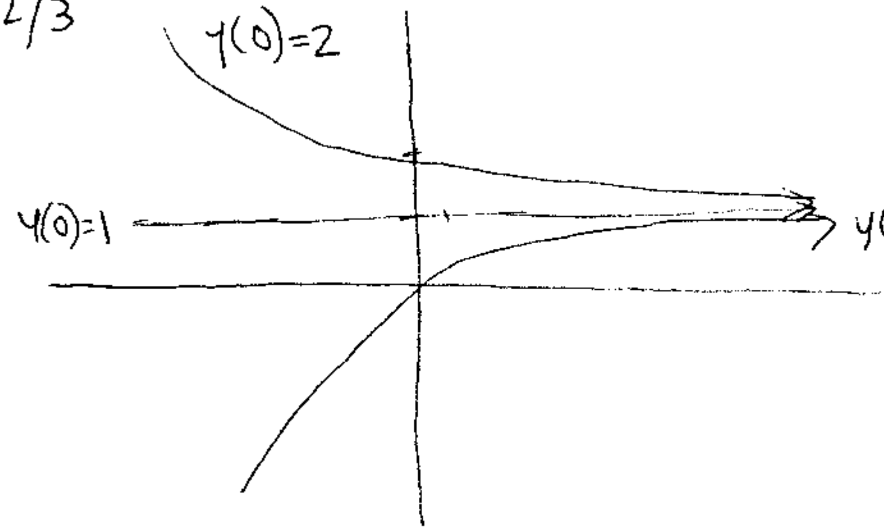
10.2/3



all approach $y = 1$

10.2/4

$y' = 1-y$

$\int \frac{dy}{1-y} = \int dx$

$-\ln(1-y) = x + c$

$\ln(1-y) = -x + c$

$1-y = e^{-x+c} = ce^{-x}$

$\boxed{y = 1 - ce^{-x}}$

a.) $y(0) = -1$

$-1 = 1 - ce^{-0}$

$-1 = 1 - c$

$c = 2$

$\boxed{y(x) = 1 - 2e^{-x}}$

b.) $y(0) = 1$

$1 = 1 - ce^{-0}$

$1 = 1 - c$

$c = 0$

$\boxed{y(x) = 1}$

c.) $y(0) = 2$

$2 = 1 - ce^{-0}$

$2 = 1 - c$

$c = -1$

$\boxed{y(x) = 1 + e^{-x}}$

10.2/7

As $x \to \infty$ the solutions to $y' = 1-y$ all become asymptotic to the line $y = 1$. Looking at the solution to the differential equation $y = 1 - ce^{x}$ we see that $\lim_{x\to\infty} y = \lim_{x\to\infty} 1 - ce^{-x} = \lim_{x\to\infty} 1 - 0 = 1$. Therefore as x approaches ∞, y approaches 0.

10.2/9a - 9

a.) $\underline{\text{IV}}$ The slope ~~fo~~ for $y' = \frac{1}{x}$ has to be + for all $x > 0$ and − for all $x < 0$

b.) $\underline{\text{VI}}$ The slope ~~of~~ for $y' = \frac{1}{y}$ has to be positive for $y > 0$ and negative for $y < 0$

9c.) V $y' = e^{-x^2}$ means the slope is always positive and y' should be greatest at $x=0$

d.) II $y' = y^2 - 1$ means the slope changes sign when crossing the lines $y = \pm 1$

10.2/13

$\frac{dy}{dx} = \sqrt{y}$ $y(0) = 1$ $0 \le x \le 4$ $h = 0.5$

$y_1 = y_0 + f(x_0, y_0)h = 1 + f(0,1) \cdot 0.5 = 1 + \sqrt{1} \cdot 0.5 = 1.5$

$y_2 = y_1 + f(x_1, y_1) h = 1.5 + \sqrt{1.5} \cdot 0.5 = 2.11$

$y_3 = y_2 + f(1, y_2) h = 2.11 + \sqrt{2.11} \cdot 0.5 = 2.84$

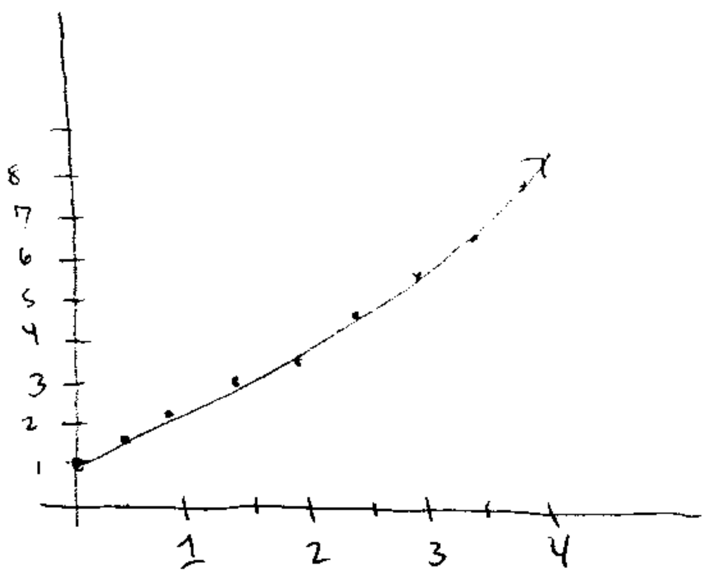
$y_4 = y_3 = f(1.5, y_3) h = 2.84 + \sqrt{2.84} \cdot 0.5 = 3.68$

$y_5 = y_4 + f(2.0, y_4) h = 3.68 + \sqrt{3.68} \cdot 0.5 = 4.64$

$y_6 = 4.64 + \sqrt{4.64} \cdot 0.5 = 5.72$

$y_7 = 5.72 + \sqrt{5.72} \cdot 0.5 = 6.92$

$y_8 = 6.92 + \sqrt{6.92} \cdot 0.5 = 8.24$



10.2/14

$\frac{dy}{dx} = x - y^2 \quad y(0) = 1 \quad 0 \le x \le 2 \quad h = 0.25$

$y_1 = y_0 + f(x_0, y_0)h = 1 + (0 - 1^2) \cdot 0.25 = 0.75$

$y_2 = 0.75 + (0.25 - 0.75^2) \cdot 0.25 = 0.67$

$y_3 = 0.67 + (0.50 - 0.67^2) \cdot 0.25 = 0.68$

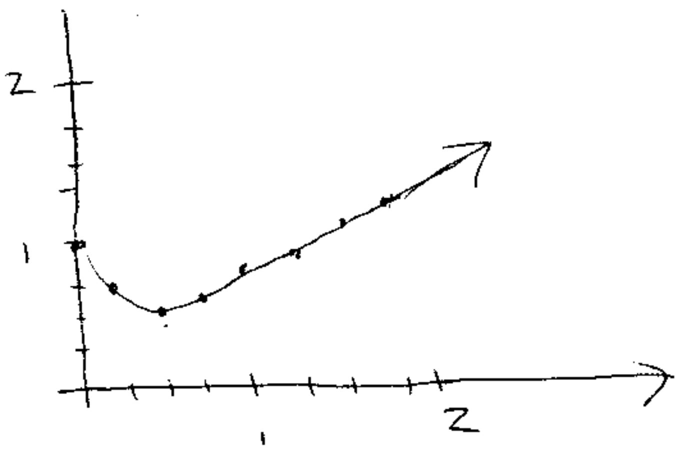
$y_4 = 0.68 + (0.75 - 0.68^2) \cdot 0.25 = 0.75$

$y_5 = 0.75 + (1.00 - 0.75^2) \cdot 0.25 = 0.86$

$y_6 = 0.86 + (1.25 - 0.86^2)\ 0.25 = 0.99$

$y_7 = 0.99 + (1.50 - 0.99^2) \cdot 0.25 = 1.124$

$y_8 = 1.12 + (1.75 - 1.12^2) \cdot 0.25 = 1.24$



10.3/6

a) $\frac{dy}{dt} = k \cdot y$

$y(t) = e^{kt} \cdot y_0$

$y(20) = 2 \cdot y_0 = y_0 \cdot e^{k \cdot 20}$

$2 = e^{k \cdot 20}$

$\ln 2 = 20k$

$k = \frac{\ln 2}{20} \Rightarrow y(t) = y_0 \cdot e^{t \ln 2/20}$

$= y_0 \cdot 2^{t/20}$

$\boxed{y = 1 \cdot 2^{t/20}}$

10.3/6 cont'd

2hrs = 120 min

c) $y(120) = 1 \cdot 2^{120/20} = 2^6 = 64$

d) $1{,}000{,}000 = e^{t\ln 2/20}$

$\ln 1{,}000{,}000 = t\ln 2/20 = 398.6$ min

10.3/7

a) $\frac{dy}{dt} = -ky$

$y = e^{-kt} \cdot y_0$

$y_0 = 5.0\times 10^7 \Rightarrow y = 5.0\times 10^7 \cdot e^{-kt}$

b) $\frac{1}{2}$ life $= 3.83$ days

$\frac{1}{2} y_0 = y_0 e^{-3.83t}$

$\frac{1}{2} = e^{-3.83k}$

$k = \frac{\ln(\frac{1}{2})}{-3.83} = 0.181$

$y(t) = 5.0\times 10^7 \cdot e^{-0.181t}$

c.) To ~~f~~ find y~~(f)~~(t) after 30 days set $t=30$

$y(30) = 5.0\times 10^7 \cdot e^{-0.181 \cdot 30}$

$= 219{,}297$

d.) $\frac{1}{10}\cdot(5.0\times 10^7) = 5.0\times 10^7 \cdot e^{-0.181t}$

$\frac{1}{10} = e^{-0.181t}$

$\ln\left(\frac{1}{10}\right) = -0.181t$

$t = 12.72$ days

10.3/17

a) $T = \frac{\ln 2}{k}$ & $\ln 2 = 0.6931$. If k is measured in %

then $T = \frac{\ln 2 \cdot 100}{k} = \frac{69.31}{k} \approx \frac{70}{k}$

b.) doubling time $= \frac{70}{1} = 70$ yrs

c.) half-life $= \frac{70}{3.5} = 20$ yrs

d.) $\frac{70}{x} = 10 \Rightarrow x = 7\%$ growth rate

10.3/23

$y = \frac{60}{5+7e^{-t}}$ follows logistic model $y = \frac{y_0 L}{y_0 + (L - y_0)e^{-kt}}$

a.) $t = 0 \Rightarrow y = \frac{60}{5+7 \cdot e^{-0}} = \frac{60}{5+7} = 5$

b.) $60 = y_0 \cdot L$

$60 = 5 \cdot L \Rightarrow L = 12$

c.) $e^{-kt} = e^{-t} \Rightarrow k = 1$

d.) when $y = \frac{1}{2} L$

$6 = \frac{60}{5+7e^{-t}} \Rightarrow 7e^{-t} = \frac{60}{6} - 5$

$7e^{-t} = 5$

$e^{-t} = \frac{5}{7}$

$t = -\ln\left(\frac{5}{7}\right) = 0.34$

e.) $\frac{dy}{dt} = \frac{1}{12} y(12 - y)$; $y_0 = 5$

10.3/19.) For carbon-14 $y(t) = y_0 e^{-0.000121t}$
Since the bones contain btw 27-30% of
the original carbon $y(t) = .27 - .30 \cdot y_0$
$y(t) = .27y_0 \Rightarrow .27 = e^{-0.000121t}$

$$t = \frac{\ln(0.27)}{-0.000121} = 10820 \approx 11{,}000 \text{ yrs ago}$$

11,000 yrs ago $\Rightarrow \approx 9000$ BC

$$y(t) = .30y_0 \Rightarrow t = \frac{\ln(.30)}{-0.000121} = 9950 \approx 10{,}000 \text{ yrs ago}$$

10,000 yrs ago $\Rightarrow \approx 8000$ BC

10.3/22.)
$$y = \frac{y_0 L}{y_0 + (L - y_0)e^{-kt}}$$

$$t = 0 \Rightarrow y = \frac{y_0 L}{y_0 + (L - y_0) \cdot 1} \Rightarrow y = \frac{y_0 L}{L} \Rightarrow y = y_0 \text{ at } t = 0$$

$y(\text{at } t=0) \approx 400$

As $t \to \infty$ $y \to L$ b/c $e^{-kt} \to 0$. In the graph it looks like y starts to level of at $y = 1000$ so $L \approx 1000$

Plugging those estimates in: $y = \dfrac{400 \cdot 1000}{400 + 600 \cdot e^{-kt}}$

use the point (1000, 1000)

$$1000 = \frac{400{,}000}{400 + 600e^{-kt}} \Rightarrow 400 + 600e^{-kt} = 400$$

Use the point (300, 700)

$$700 = \frac{400 \cdot 1000}{400 + 600 \cdot e^{-k \cdot 300}}$$

$$e^{-k \cdot 300} = 0.2857$$

$$k = \frac{\ln 0.2857}{-300} = +0.0042$$

9/2

Check that $y = A\cos t + B\sin t$ is a soln to $y'' + y = 0$

$y' = A \cdot -\sin t + B\cos t$

$y'' = A \cdot -\cos t + B \cdot -\sin t = -A\cos t - B\sin t$

$y'' + y = -A\cos t - B\sin t + A\cos t + B\sin t = 0$ ✓

9/4

$y = A\cos\omega t + B\sin\omega t$ is a soln to $y'' + 16y = 0$

$y(0) = 2$

$y(\frac{\pi}{8}) = 3$

general soln of $\frac{d^2y}{dt^2} + \omega^2 y = 0$ is $y(t) = C_1\cos\omega(t) + C_2\sin(\omega t)$

$\omega = \sqrt{16} = 4$

$y = A\cos 4t + B\sin 4t$

Plug in $y(0) = 2$

$y(0) = 2 = A \cdot \cos 0 + B\sin 0 = A$

$A = 2$

Plug in $y(\frac{\pi}{8}) = 3$

$y(\frac{\pi}{8}) = 3 = A \cdot \cos(\frac{\pi}{2}) + B \cdot \sin(\frac{\pi}{2})$

$= B$

$\Rightarrow B = 3$

$y(t) = 2\cos 4t + 3\sin 4t$

9/8

a) $y'' + 4y = 0$

general solution is of the form

$y(t) = c_1 \cos \omega t + c_2 \sin \omega t$

$\omega = \sqrt{4} = 2$

$y(t) = c_1 \cos 2t + c_2 \sin 2t$

bi) $y(0) = 5 \quad y'(0) = 0$

$y'(t) = c_1 \cdot 2 \cdot -\sin 2t + c_2 \cdot 2 \cdot \cos 2t$

$= -2c_1 \sin 2t + 2c_2 \cos 2t$

$y(0) = 5 \Rightarrow c_1 \cos 2\cdot 0 + c_2 \sin 2 \cdot 0 = 5$

$c_1 \cdot 1 + 0 = 5$

$c_1 = 5$

$y'(0) = 0 \Rightarrow -2c_1 \cdot \sin 2 \cdot 0 + 2c_2 \cos 2 \cdot 0 = 0$

$0 + 2c_2 \cdot 1 = 0$

$c_2 = 0$

$y(t) = 5 \cos 2t$

bii) $y(0) = 0 \quad y'(0) = 10$

$y(0) = 0 \Rightarrow c_1 \cos 2 \cdot 0 + c_2 \sin 2 \cdot 0 = 0$

$c_1 \cdot 1 = 0$

$c_1 = 0$

$y'(0) = 10 \Rightarrow 2c_2 \cos 2 \cdot 0 - 2 \cdot c_1 \sin 2 \cdot 0 = 10$

$2 \cdot c_2 \cdot 1 = 10$

$c_2 = 5$

$y(t) = 5 \sin 2t$

biii.) $y(0) = 5, \; y'(0) = 5$

$y(0) = 5 \Rightarrow c_1 \cdot 1 = 5 \Rightarrow c_1 = 5$

$y'(0) = 5 \Rightarrow c_2 \cdot 2 = 5 \Rightarrow c_2 = \frac{5}{2}$

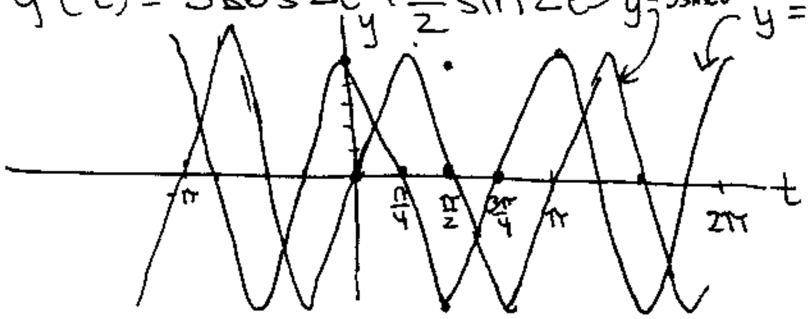
$y(t) = 5\cos 2t + \frac{5}{2} \sin 2t \rightarrow A = \sqrt{5^2 + \frac{5}{2}^2} = 5.6$

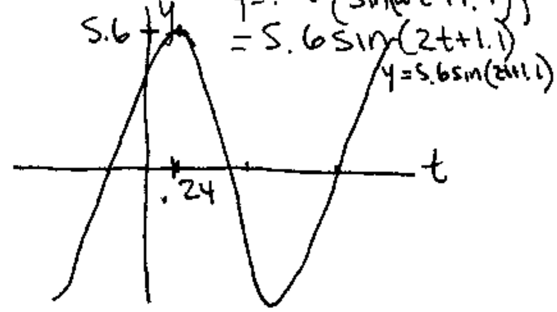
$\phi = \tan^{-1}\left(\frac{5}{\frac{5}{2}}\right) = \tan^{-1} 2 = 1.1$

$y = 5.6\left(\sin(\omega t + 1.1)\right)$

$= 5.6 \sin(2t + 1.1)$

c.)





9/9

a.) $x'' + x = 0$

$\Rightarrow$ soln's of form $c_1 \sin t + c_2 \cos t$ $(\omega = 1)$

b.) $x'' + 4x = 0$

$\Rightarrow$ soln's of form $c_1 \sin 2t + c_2 \cos 2t$ $(\omega = 2)$

c.) $x'' + 16x = 0$

$\Rightarrow$ soln's of form $c_1 \sin 4t + c_2 \cos 4t$ $(\omega = 2)$

Period of a is greatest, period of c is least

$\Rightarrow a = \underline{\text{II}}$, $c = \underline{\text{III}}$, I & $\underline{\text{IV}}$ both have same period

$\Rightarrow$ both soln's for b

Find eqn's for the graphs:

(I) $y(0) = 0 \Rightarrow c_1 \cdot 0 + c_2 \cdot \cos 0 = 0$

$c_2 = 0$

$y(t) = c_1 \sin 2t$

Amplitude $A = 2 \Rightarrow y(t) = 2\sin 2t$

(II) $y(0) = 0 \Rightarrow c_2 = 0$

$y(t) = c_1 \sin t$

Amplitude $= 1$, sine curve flipped

$\Rightarrow y(t) = -\sin t$

(III) $y(0) =$ max of y

$\Rightarrow c_1 = 0 \quad y = c_2 \cos 4t$

Amplitude $= 1 \Rightarrow y = \cos 4t$

(IV) $y(0) = 0$

$\Rightarrow c_2 = 0$

$y(t) = c_1 \sin 2t$

amplitude $= 3$, sine curve flipped

$y(t) = -3\sin 2t$

9/10

a.) shortest period $\Rightarrow T=\frac{2\pi}{\omega}$ is the smallest

$\Rightarrow \omega$ is the biggest

$s''+6s=0 \Rightarrow \omega=\sqrt{6} \Rightarrow$ iii has shortest period

b.) amplitude $= \sqrt{c_1^2+c_2^2}$

Since $s'(0)=0$ for all the eqn's all of them are of the form

$y(t)=c_1\cos\omega t$ with $c_1=s(0)$

Therefore, the eqn w/ the greatest $s(0)$ has the greatest amplitude $\Rightarrow$ iv has the shortest amplitude

c.) longest period $\Rightarrow T=\frac{2\pi}{\omega}$ is the largest

$\Rightarrow \omega$ is the smallest

iv has $\omega=\sqrt{\frac{1}{6}} \Rightarrow$ iv has the longest period

d.) velocity $=\frac{dy}{dx} = c_1\cdot\omega\cdot -\sin\omega t$

max velocity $= c_1\cdot\omega$

iii has max velocity $4\sqrt{6}$ which is the greatest of the lot $\Rightarrow$ iii has the largest max velocity

9/14

Write $7\sin\omega t+24\cos\omega t$ in the form

$A\sin(\omega t+\phi)$

$\tan\phi = \frac{c_1}{c_2} = \frac{7}{24} \Rightarrow \phi=\tan^{-1}\left(\frac{7}{24}\right)=0.28\,\text{rad}$

$A=\sqrt{c_1^2+c_2^2}=\sqrt{7^2+24^2}=25$

$= 25\sin(\omega t+0.28)$