Exam Two Review — Integration!! Part 1 — Fall '99 Math 1B

Topics covered since last exam:

Integration Techniques:

contained in this review package:
1. Integration by Substitution
2. Integration by Parts
3. Partial Fractions
4. Improper Integrals

Integration Applications:

in Exam 2 Review part 2:
5. Volume Integrals
6. Arc Length
7. Surface Area
8. Work Integrals
9. Fluid Force

1 Substitution -

very common technique - often have to do a simple substitution before you apply another integration technique

Guidelines:
- try to sub for the denominator, (or most of it)
- typically sub for functions within other functions
- often can sense an appropriate function to sub for by seeing its derivative floating around

Note!
- Don't forget to "unsub" at end of integration
- for Definite Integrals - do out whole integral, then evaluate result at the endpoints

Example 1 $\displaystyle\int_0^3 \frac{7x}{1+x^2}\,dx$

Example 2 $\displaystyle\int \cos x\, e^{\sin x}\,dx$

① Sub. con't

Setting up for an integration by parts...

Example 3 $\displaystyle\int x^5 \sin(x^3)\,dx$

Trick substitutions – sometimes have to do a bit of algebraic manipulation to solve for extra "stuff" hanging around –

Example 4 $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx$

Trickier sub! if you see a function within a function and there's nothing else to do, try subbing along with more manipulation

Example 5 $\displaystyle\int \sin(\sqrt{x})\,dx$

Beware!! Check denominators for blow-ups
(You know that there are improper integrals out there!)

Example 6 $\displaystyle\int_0^1 \frac{1}{1-x}\,dx$

② Integration by Parts

remember the basic set-up: $\displaystyle\int u\,dv =$

- you might have to do a simple substitution first
- for definite integrals, work through whole integral first, then evaluate result at endpoints

Example 7 $\displaystyle\int_0^{\pi} x^2 \sin x\,dx$

② Int. by Parts continued

Example 8 $\int e^x \sin(e^x)\,dx$

Example 9 $\int e^{3x} \sin(e^x)\,dx$

Example 10 $\int \ln x\,dx$

"trick" integral

Example 11 $\int x^n \ln x\,dx$ (assume $n \neq -1$)

Example 12 $\int_1^e x^{-1} \ln x\,dx$

② Integration by Parts continued

<u>Example</u> 13 $\int_0^1 x \ln(x+3)\,dx$

③ Partial Fractions

To evaluate $\int \frac{P(x)}{D(x)}\,dx$ ($P(x)$, $D(x)$ just polynomials in x)

0) check $\underbrace{\text{degree}(P(x))}_{\text{numerator}} < \underbrace{\text{degree}(D(x))}_{\text{denominator}}$

— if not do long division first

1) Factor denominator into irreducible terms
2) Write as $\frac{A}{(\text{Factor }1)} + \frac{B}{(\text{Factor }2)} + \text{etc.}$
3) Cross multiply, equate numerators to find $A, B, \ldots$
4) Integrate resulting fractions

<u>Example</u> 14 $\int \frac{6-x}{2x^2-x-15}\,dx$

Example 15 $\displaystyle\int \frac{x+1}{(x^2-2x)(x+3)}\,dx$

④ Improper Integrals!

$\boxed{1^{st}\text{ kind}}$ $\displaystyle\int_a^{\infty} f(x)\,dx =$ and $\displaystyle\int_{-\infty}^{b} f(x)\,dx =$

Example 16 Evaluate $\displaystyle\int_1^{\infty} \frac{6x^2+4}{(x^3+2x)^2}\,dx$

Example 17 Evaluate $\displaystyle\int_{30}^{\infty} \frac{\ln x}{x}\,dx$

$\boxed{2^{nd}\text{ kind}}$ if $f(x)$ is undefined at some point $x=d$

then $\displaystyle\int_a^{d} f(x)\,dx =$ and $\displaystyle\int_d^{b} f(x)\,dx =$

what if $a<d<b$ then $\displaystyle\int_a^{b} f(x)\,dx =$

(4) Improper Integrals continued

Example 18 Evaluate $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\, dx$

Example 19 Evaluate $\int_{-2}^{2} x^{-2}\, dx$

Note: Improper Integrals involves familiarity with taking limits:

Know $\lim_{x\to\infty} e^x = \infty = \lim_{x\to\infty} \ln x$

while $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to 0} \ln x = -\infty$

Also be familiar with L'Hopital's Rule:

a) $\lim_{x\to\infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$

b) $\lim_{x\to\infty} x^{-3} e^x = \lim_{x\to\infty} \frac{e^x}{x^3} = ?$

c) so $\lim_{x\to\infty} x^{100} e^{-x} = ?$

Answers to Examples

Example 1 $\int_0^3 \frac{7x}{1+x^2}\,dx$ sub $u=1+x^2$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$ get $\int_{x=0}^{x=3} \frac{7(\frac{1}{2}du)}{u} = \frac{7}{2}\ln|u|\Big|_{x=0}^{x=3}$

$$= \frac{7}{2}\ln|1+x^2|\Big|_0^3 = \frac{7}{2}\ln 10 - \frac{7}{2}\ln 1$$
$$= \frac{7}{2}\ln 10$$

Example 2 $\int \cos x\, e^{\sin x}\,dx$ $u = \sin x$, $du = \cos x\,dx$ get $\int e^u\,du = e^u + c$

$$= e^{\sin x} + C$$

Example 3 $\int x^5 \sin(x^3)\,dx$ sub $u = x^3$, $du = 3x^2dx$, $\frac{1}{3}du = x^2dx$ get $\int u\sin(u)\cdot\frac{1}{3}du$

$$= \int x^3\cdot x^2\cdot \sin(x^3)\,dx \qquad = \frac{1}{3}\int u\sin(u)\,du$$

now do int. by parts–
change variables $\frac{1}{3}\int w\sin(w)\,dw$ so you don't get confused!
just remember that $w = u = x^3$

now let $u = w$ $\quad du = dw$
$dv = \sin(w)\,dw$ $\quad v = -\cos(w)$

get $\frac{1}{3}\left(-w\cos(w) - \int -\cos(w)\,dw\right)$ (*)

$$= \frac{1}{3}\left(-w\cos(w) + \sin(w)\right) + C$$
$$= \frac{1}{3}\left(-x^3\cos(x^3) + \sin(x^3)\right) + C$$

Example 4 well try $u = 1+x^2$, so $du = 2x\,dx$, also $x^2 = u-1$
$\frac{1}{2}du = x\,dx$

so in total have $\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{x^2\cdot x\,dx}{\sqrt{1+x^2}} = \int \frac{(u-1)}{\sqrt{u}}\frac{1}{2}du$

$$= \int \frac{1}{2}\left(u^{1/2} - u^{-1/2}\right)du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right) + C$$
$$= \frac{1}{3}(1+x^2)^{3/2} - (1+x^2)^{1/2} + C$$

Example 5 $\int \sin(\sqrt{x})\,dx$ well, nothing else to do, try $u = \sqrt{x}$
Then $du = \frac{1}{2}x^{-1/2}dx$, okay, now note
this implies $2x^{1/2}du = dx$ (multiply both sides by $2x^{1/2}$)
$= 2u\,du$!

Finally, then, you've got

$$\int \sin(\sqrt{x})\,dx = \int \sin(u)\cdot 2u\,du = 2\int u\sin(u)\,du$$

which, by parts, yields $2\left(-\sqrt{x}\cos(\sqrt{x}) + \sin(\sqrt{x})\right) + C$
by (*)

Answers cont

Example 6 $\int_0^1 \frac{1}{1-x}\,dx$ blows up at $x=1$, so really should be set up as $\lim_{t\to 1^-} \int_0^t \frac{1}{1-x}\,dx$

now $\int \frac{1}{1-x}dx$, $\begin{array}{l} u=1-x \\ du=-dx \end{array}$ $= \int \frac{-du}{u} = -\ln|u|+C$
$= -\ln|1-x|+C$,

so $\lim_{t\to 1^-}\left(\int_0^t \frac{1}{1-x}dx\right) = \lim_{t\to 1^-}\left(-\ln|1-x|\Big|_0^t\right)$

$= \lim_{t\to 1^-}\left(-\ln|1-t| - \underbrace{(-\ln|1|)}_{=0}\right) = \lim_{t\to 1^-}\left(-\ln|1-t|\right)$ diverge

so answer: integral diverges

Example 7 $\int_0^\pi x^2 \sin x\,dx$ let $\begin{array}{ll} u=x^2 & dv=\sin x\,dx \\ du=2x\,dx & v=-\cos x \end{array}$

so $\int x^2\sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx = -x^2\cos x + 2\int x\cos x\,dx$

now $\int x\cos x\,dx$: $\begin{array}{ll} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x \end{array}$ so $= x\sin x - \int \sin x\,dx$
$= x\sin x - (-\cos x)$

so in total, get $\left(-x^2\cos x + 2\left[x\sin x + \cos x\right]\right)\Big|_0^\pi$

$= -\pi^2(-1) + 2(\pi\cdot 0 + (-1)) - (0 + 2(0+1))$

$= \pi^2 - 2 - 2 = \pi^2 - 4$

Example 8 $\int e^x \sin(e^x)\,dx$ sub $\begin{array}{l} u=e^x \\ du=e^x dx \end{array}$ $\Rightarrow \int \sin(u)\,du = -\cos u + C$
$= -\cos(e^x) + C$

Example 9 $\int e^{3x}\sin(e^x)\,dx$ again sub $\begin{array}{l} u=e^x \\ du=e^x dx \end{array}$ note $e^{3x} = (e^x)^3 = (e^x)^2\cdot e^x$

so $= \int u^2 \sin(u)\,du$

already know $\int x^2 \sin x\,dx = -x^2\cos x + 2x\sin x + 2\cos x + C$

so get $-e^{2x}\cos(e^x) + 2e^x\sin(e^x) + 2\cos(e^x) + C$

Example 10 $\int \ln x\,dx$ $\begin{array}{ll} u=\ln x & dv=dx \\ du=\frac{1}{x}dx & v=x \end{array}$ get $x\ln x - \int x\frac{1}{x}dx$
$= x\ln x - \int dx = x\ln x - x + C$

Example 11 $\int x^n \ln x \, dx$ $\quad u = \ln x \quad dv = x^n dx$
$\quad du = \frac{1}{x} dx \quad v = \frac{x^{n+1}}{n+1}$

$$\Rightarrow \frac{x^{n+1}}{n+1} \ln x - \int \frac{x^{n+1}}{n+1} \frac{1}{x} \, dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{1}{n+1} \int x^n \, dx$$

$$= \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

Example 12 $\int_1^e x^{-1} \ln x \, dx$ $\quad$ sub $u = \ln x$
$du = \frac{1}{x} dx = x^{-1} dx$

$$\text{so } \int x^{-1} \ln x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

$$\text{so } \int_1^e x^{-1} \ln x \, dx = \frac{(\ln x)^2}{2} \Big|_1^e = \frac{(\ln e)^2}{2} - \frac{(\ln 1)^2}{2} = \frac{1}{2}$$

Example 13 $\int_0^1 x \ln(x+3) \, dx$ $\quad u = \ln(x+3) \quad dv = x \, dx$
$\quad du = \frac{1}{x+3} dx \quad v = \frac{x^2}{2}$

$$\Rightarrow \frac{x^2}{2} \ln(x+3) - \frac{1}{2} \int \frac{x^2}{x+3} \, dx$$

do not need partial fractions, just do long division

$$\begin{array}{r} x - 3 \\ x+3 \overline{) x^2 + 0 \cdot x + 0} \\ \underline{x^2 + 3x} \\ -3x + 0 \\ \underline{-3x - 9} \\ 9 \end{array}$$

$$\text{so } \frac{x^2}{x+3} = x - 3 + \frac{9}{x+3}$$

$$\text{so } \frac{1}{2} \int \frac{x^2}{x+3} \, dx = \frac{1}{2} \int \left(x - 3 + \frac{9}{x+3}\right) dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - 3x + 9 \ln|x+3|\right)$$

in total get

$$\left[\frac{x^2}{2} \ln(x+3) - \frac{1}{2}\left(\frac{x^2}{2} - 3x + 9 \ln|x+3|\right)\right]_0^1$$

$$= \left(\frac{1}{2} \ln 4 - \frac{1}{2}\left(\frac{1}{2} - 3 + 9 \ln 4\right)\right) - \left(0 - \frac{1}{2}(0 - 30 + 9 \ln 3)\right)$$

$$= -4 \ln 4 + \frac{5}{4} + \frac{9}{2} \ln 3$$

Example 14 $= \int \frac{6-x}{(x-3)(2x+5)} \, dx$ $\quad \frac{6-x}{(x-3)(2x+5)} = \frac{A}{x-3} + \frac{B}{2x+5}$

so $(2x+5)A + (x-3)B = 6 - x$

sub $x = 3 \Rightarrow 11A = 3$ so $A = 3/11$

sub $x = -\frac{5}{2} \Rightarrow -\frac{11}{2} B = \frac{17}{2} \Rightarrow B = -17/11$

so integral $= \int \frac{3/11}{x-3}\,dx + \int \frac{-17/11}{2x+5}\,dx = \int \frac{3/11}{x-3}\,dx - 17/11\left(\frac{1}{2}\right)\int \frac{1}{x+5/2}$

$= 3/11 \ln|x-3| - (17/11)\left(\frac{1}{2}\right) \ln|x+5/2| + C$

$= 3/11 \ln|x-3| - 17/22 \ln|x+5/2| + C$

Example 15 $= \int \frac{x+1}{x(x-2)(x+3)}\,dx \qquad \frac{x+1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$

then $(x-2)(x+3)A + x(x+3)B + x(x-2)C = x+1$

sub $x=2 \Rightarrow 10B = 3 \quad B = 3/10$

sub $x=0 \Rightarrow -6A = 1 \quad A = -1/6$

sub $x=-3 \Rightarrow 15C = -2 \quad C = -2/15$

so integral $= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C$

Example 16 $= \lim_{N\to\infty}\left(\int_1^N \frac{6x^2+4}{(x^3+2x)^2}\,dx\right)$ now sub $u = x^3+2x \quad du = (3x^2+2)$

then $\int \frac{6x^2+4}{x^3+2x}\,dx = \int \frac{2\,du}{u^2} = -\frac{2}{u} + C = \frac{-2}{x^3+2x} + C$

so original improper integral $= \lim_{N\to\infty}\left(\frac{-2}{x^3+2x}\Big|_1^N\right)$

$= \lim_{N\to\infty}\left(\underbrace{\frac{-2}{N^3+2N}}_{0} - \left(-\frac{2}{3}\right)\right) = \frac{2}{3}$

Example 17 $\int_{30}^{\infty} \frac{\ln x}{x} = \lim_{N\to\infty}\left(\int_{30}^N \frac{\ln x}{x}\,dx\right)$ sub $u = \ln x \quad du = \frac{1}{x}\,dx$

then $\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2}$

$= \frac{(\ln x)^2}{2}$

so $= \lim_{N\to\infty}\left(\frac{(\ln x)^2}{2}\Big|_{30}^N\right) = \lim_{N\to\infty}\left(\underbrace{\frac{(\ln N)^2}{2}}_{\infty} - \frac{(\ln 30)^2}{2}\right) \Rightarrow$ diverges

Example 18 $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx$ sub $u = \sin x$, $du = \cos x\,dx$, so $\int \frac{\cos x}{\sqrt{\sin x}}\,dx = \int \frac{du}{\sqrt{u}}$

improper as $\sin x = 0$ when $x = 0$

$= 2u^{1/2} + C = 2\sqrt{\sin x} + C$

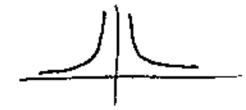
so $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{N\to 0^+} \int_N^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx$

$= \lim_{N\to 0^+}\left[2\sqrt{\sin x}\Big|_N^{\pi/2}\right] = \lim_{N\to 0^+}\left[2 - \underbrace{2\sqrt{\sin N}}_{0}\right]$

$= 2$

Answers continued

Example 19 $\int_{-2}^{2} x^{-2}\,dx \rightarrow$ improper at $x=0$

(note graph: of $x^{-2} = \frac{1}{x^2}$ )

so break into two improper integrals:

$$= \lim_{N\to 0^-} \int_{-2}^{N} x^{-2}\,dx + \lim_{M\to 0^+} \int_{M}^{2} x^{-2}\,dx = \lim_{N\to 0^-} \left[-\frac{1}{x} \Big|_{-2}^{N} \right] + \lim_{M\to 0^+} \left[-\frac{1}{x} \Big|_{M}^{2} \right]$$

$$= \lim_{N\to 0^-} \left[\underbrace{-\frac{1}{N}}_{\infty} - \left(-\frac{1}{-2} \right) \right] + \lim_{M\to 0^+} \left[-\frac{1}{2} - \underbrace{\left(-\frac{1}{M} \right)}_{\infty} \right]$$

so **diverges**

also b) $\lim_{x\to\infty} \frac{e^x}{x^3} = \lim_{x\to\infty} \frac{e^x}{3x^2} = \lim_{x\to\infty} \frac{e^x}{6x} = \lim_{x\to\infty} \frac{e^x}{6} = \infty$

and c) $\lim_{x\to\infty} x^{100} e^{-x} = \lim_{x\to\infty} \frac{x^{100}}{e^x} = \lim_{x\to\infty} \frac{100x^{99}}{e^x} = \ldots$

$\ldots = \lim_{x\to\infty} \frac{(100)!}{e^x} = 0$