

**First Exam**

- Do not open this exam booklet until you are directed to do so.
- You have 120 minutes to earn 80 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

| Problem | Points | Grade |
|---------|--------|-------|
| 1       | 15     | 15    |
| 2       | 10     | 10    |
| 3       | 10     | 10    |
| 4       | 10     | 10    |
| 5       | 10     | 10    |
| 6       | 15     | 15    |
| 7       | 10     | 10    |
| Total   | 80     | 80    |

Please circle your section:

|                              |                                |                                |                             |                                 |                                |
|------------------------------|--------------------------------|--------------------------------|-----------------------------|---------------------------------|--------------------------------|
| MWF 10:00<br>Brian<br>Conrad | MWF 10:00<br>Andy<br>Engelward | MWF 10:00<br>Eric<br>Towne     | MWF 11:00<br>Noam<br>Elkies | MWF 11:00<br>Robert<br>Pollack  | MWF 12:00<br>Andy<br>Engelward |
|                              | TTh 10:00<br>Tomas<br>Klenke   | TTh 10:00<br>Joel<br>Rosenberg | TTh 10:00<br>Eric<br>Towne  | TTh 11:30<br>Heather<br>Russell |                                |

1. (15 pts)

1a. Determine whether each of the following series converges or diverges. Justify your answers.

$$(i) \sum_{k=1}^{\infty} \left(1 - \frac{1}{k^2}\right)$$

Since  $\lim_{k \rightarrow \infty} \left(1 - \frac{1}{k^2}\right) = 1 \neq 0$ , the series diverges by the divergence test.

$$(ii) \sum_{n=0}^{\infty} \frac{n^2 + 1}{n!}$$

RATIO TEST:  $\lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2 + 1}{(n+1)!}}{\frac{n^2 + 1}{n!}} = \lim_{n \rightarrow \infty} \frac{1}{n+1} \cdot \frac{(n+1)^2 + 1}{n^2 + 1} = 0 < 1$

so the series CONVERGES

$$(iii) \sum_{k=1}^{\infty} \frac{\cos^2 k}{k^2 + 1}$$

Since  $\frac{\cos^2 k}{k^2 + 1} < \frac{1}{k^2 + 1} < \frac{1}{k^2}$ , the series converges by comparison with the p-series  $\sum_{k=1}^{\infty} \frac{1}{k^2}$

1b. For what real values of  $a$  do the following series converge? Explain your answers.

$$(i) \sum_{k=1}^{\infty} a$$

The sequence  $a, a, a, a, \dots$  converges to  $a$ , so the series diverges if  $a \neq 0$  by the divergence test.

If  $a = 0$ , the series converges.

$$(ii) \sum_{k=1}^{\infty} (-1)^k k^a$$

If  $a > 0$ ,  $\lim_{k \rightarrow \infty} k^a = \infty$ , so the series diverges by divergence test.

If  $a = 0$ ,  $\sum_{k=1}^{\infty} (-1)^k = -1 + 1 - 1 + 1 - \dots$  diverges.

If  $a < 0$ , then  $\lim_{k \rightarrow \infty} k^a = 0$  and  $1 > 2^a > 3^a > 4^a > \dots$ ,

so the series converges by the alternating series test.

Conclusion: The series converges for  $\boxed{a < 0}$  only.

2. (10 pts) For each power series below, give the radius and interval of convergence. Be sure to check convergence at the endpoints.

a. 
$$\sum_{k=0}^{\infty} \frac{k!(x-3)^k}{2^k}$$

$$\lim_{k \rightarrow \infty} \frac{\frac{(k+1)! |x-3|^{k+1}}{2^{k+1}}}{\frac{k! |x-3|^k}{2^k}} = \lim_{k \rightarrow \infty} \frac{k+1}{2} \cdot |x-3| = \infty, \text{ for } x \neq 3.$$

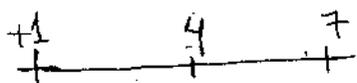
So the radius of convergence is  $R=0$ , the series converges only for  $x=3$ .  
by the ratio test

b. 
$$\sum_{k=1}^{\infty} \frac{(x-4)^k}{k \cdot 3^k}$$

$$\lim_{k \rightarrow \infty} \frac{\frac{|x-4|^{k+1}}{(k+1)3^{k+1}}}{\frac{|x-4|^k}{k \cdot 3^k}} = \lim_{k \rightarrow \infty} \frac{k}{k+1} \cdot \frac{|x-4|}{3} = \frac{|x-4|}{3}$$

By the ratio test, the series  $\begin{cases} \text{converges for } \frac{|x-4|}{3} < 1, |x-4| < 3 \\ \text{diverges for } \frac{|x-4|}{3} > 1, |x-4| > 3. \end{cases}$

So  $R=3$  is the radius of convergence.



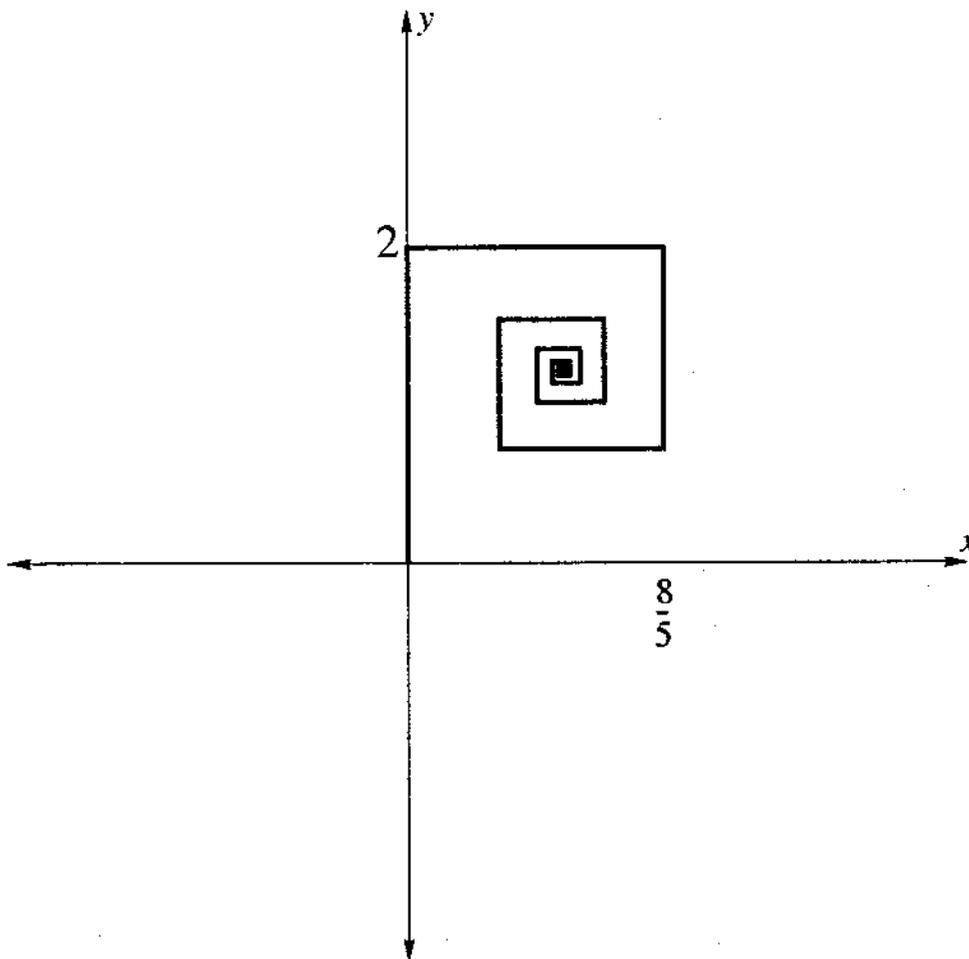
Endpoints:  $x=1$  :  $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$  converges (alternating harmonic series)

$x=7$  :  $\sum_{k=1}^{\infty} \frac{3^k}{k \cdot 3^k} = \sum_{k=1}^{\infty} \frac{1}{k}$  diverges (harmonic)

So the interval of convergence is  $[1, 7)$ .

3. (10 pts) I want to draw a spiral, but since I am not that good at drawing curves, I've decided to draw a "spiral with corners" instead, as follows:

I start at the origin  $(0,0)$  of the  $xy$ -plane, and draw a segment 2 units long, vertically upwards. Then I turn right by  $90^\circ$ , and draw a segment  $\frac{4}{5}$  as long as the first segment. Then I make another  $90^\circ$  right turn, and draw a segment  $\frac{4}{5}$  as long as the second segment. I continue this way, and get my spiral:



(Question continues on next page)

a. What is the total distance my pen will eventually travel?

$$\text{TOTAL DISTANCE: } L = 2 + 2 \cdot \frac{4}{5} + 2 \cdot \left(\frac{4}{5}\right)^2 + 2 \cdot \left(\frac{4}{5}\right)^3 + \dots = 2 \cdot \frac{1}{1 - \frac{4}{5}} = 10$$

(geometric series)

b. Find the coordinates  $(x_0, y_0)$  of the point  $P$  in the  $xy$ -plane at which I will eventually end up. (Hint: consider the  $x$ -coordinate after each segment is drawn, and then consider the  $y$ -coordinate after each segment is drawn.)

$$y_0 = 2 - 2 \cdot \left(\frac{4}{5}\right)^2 + 2 \cdot \left(\frac{4}{5}\right)^4 - 2 \cdot \left(\frac{4}{5}\right)^6 + \dots$$

$$= 2 \left[ 1 + \left(-\frac{16}{25}\right) + \left(\frac{16}{25}\right)^2 - \left(\frac{16}{25}\right)^3 + \dots \right]$$

$$= 2 \cdot \frac{1}{1 + \frac{16}{25}} = \frac{50}{41}$$

Geometric series with  
 $r = -\frac{16}{25}$ :

$$x_0 = 2 \cdot \frac{4}{5} - 2 \cdot \left(\frac{4}{5}\right)^3 + 2 \cdot \left(\frac{4}{5}\right)^5 - 2 \cdot \left(\frac{4}{5}\right)^7 + \dots$$

$$= \frac{8}{5} \left( 1 - \frac{16}{25} + \left(\frac{16}{25}\right)^2 - \left(\frac{16}{25}\right)^3 + \dots \right) = \frac{8}{5} \cdot \frac{1}{1 + \frac{16}{25}} = \frac{40}{41}$$

$$\text{So } (x_0, y_0) = \left( \frac{40}{41}, \frac{50}{41} \right)$$

4. (10 pts)

a. Find the second degree Taylor polynomial for  $f(x) = \sqrt{x}$  centered at  $x = 4$ .

$$f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \quad f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$$

$$f(4) = \sqrt{4} = 2, \quad f'(4) = \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{1}{4}, \quad f''(4) = -\frac{1}{4} \cdot \frac{1}{(\sqrt{4})^3} = -\frac{1}{32}$$

$$P_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

b. Use your answer to a to give an approximation to  $\sqrt{3.9}$ . (You need not simplify your answer.)

$$\sqrt{3.9} \approx P_2(3.9) = 2 - \frac{1}{4} \cdot (.1) - \frac{1}{64} \cdot (.1)^2$$

c. If you wanted to approximate  $\sqrt[3]{31}$  by using a Taylor series for  $\sqrt[3]{x}$ , where would you center your Taylor polynomial? Explain your reasoning briefly. Do not attempt to approximate  $\sqrt[3]{31}$ ; just say how you would do it.

I would center the Taylor polynomial at  $x=27$ , because  $f(27), f'(27), f''(27), \dots$  are all easily computable ratios of integers.

$$f(27) = \sqrt[3]{27} = 3$$

$$f'(27) = \frac{1}{3} x^{-\frac{2}{3}} \Big|_{x=27} = \frac{1}{3} \cdot \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{27} \text{ etc.}$$

Thus the Taylor polynomial about  $x=27$  has rational coefficients.

5. (10 pts) Show that if we try to compute  $\sqrt[3]{1.06}$  by approximating the function

$$g(x) = \sqrt[3]{1+x}$$

by the polynomial  $1 + \frac{1}{3}x$ , then the error will be less than .0005.

Haven't done this in the course (binomial series)...

but you can still work this out using the Maclaurin series for  $g(x) = \sqrt[3]{1+x} = (1+x)^{1/3}$

$$\begin{aligned} \text{so } g'(x) &= \frac{1}{3}(1+x)^{-2/3} & \text{and } g(0) &= 1 \\ g''(x) &= -\frac{2}{9}(1+x)^{-5/3} & g'(0) &= \frac{1}{3} \\ g'''(x) &= \frac{10}{27}(1+x)^{-8/3} & g''(0) &= -\frac{2}{9} \\ & & g'''(0) &= \frac{10}{27} \end{aligned}$$

From here on, the signs alternate

$$\begin{aligned} \text{Maclaurin series for } \sqrt[3]{1+x} \text{ is } & 1 + \frac{1}{3}x - \frac{\left(\frac{2}{9}\right)}{(2!)}x^2 + \frac{\left(\frac{10}{27}\right)}{(3!)}x^3 - \dots \\ & = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \dots \end{aligned}$$

it's an alternating series after first two terms for  $x > 0$ ,  
(with decreasing terms if  $x = .06$ , where it converges)

$$\text{when } x = .06 \text{ then } g(.06) = \sqrt[3]{1+.06} = \sqrt[3]{1.06}$$

$$\approx 1 + \frac{1}{3}(.06), \text{ with}$$

$$|\text{error}| < \frac{1}{9}(.06)^2 = \frac{1}{9} \cdot \left(\frac{6}{100}\right)^2$$

$$= \frac{36}{9} \cdot \frac{1}{10,000} = .0004$$

$$< .0005$$

6. (15 pts) Let  $f(x)$  be given by the power series

$$f(x) = \sum_{k=1}^{\infty} \frac{(x-2)^k}{\sqrt{k}}$$

a. Determine the radius of convergence and the interval of convergence of this series. Be sure to check convergence at the endpoints.

$$\lim_{k \rightarrow \infty} \frac{|x-2|^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{|x-2|^k} = \lim_{k \rightarrow \infty} \frac{\sqrt{k}}{\sqrt{k+1}} \cdot |x-2| = |x-2|.$$

By the ratio test, the series converges for  $|x-2| < 1$ , diverges for  $|x-2| > 1$ .

So the radius of convergence is 1.  $|x-2| < 1$  means  $-1 < x-2 < 1$

Endpoints:  $x=1$ :  $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$  converges (alternating series test, since  $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}} = 0$ , and  $\frac{1}{\sqrt{k}} > \frac{1}{\sqrt{k+1}}$ )

$x=3$ :  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$  diverges ( $p$ -series,  $p = \frac{1}{2}$ )

So the interval of convergence is  $[1, 3)$

b. Write out the first four nonzero terms of the power series for the derivative  $\frac{df}{dx}$ .

What are the radius and interval of convergence of the power series for  $\frac{df}{dx}$ ?

$$\frac{df}{dx} = \sum_{k=1}^{\infty} \frac{k(x-2)^{k-1}}{\sqrt{k}} = \sum_{k=1}^{\infty} \sqrt{k}(x-2)^{k-1} = 1 + \sqrt{2}(x-2) + \sqrt{3}(x-2)^2 + \sqrt{4}(x-2)^3 + \dots$$

The radius of convergence is the same as that for  $f(x)$ , that is  $R=1$ .

Endpoints:  $x=1$ :  $\sum_{k=1}^{\infty} (-1)^k \sqrt{k}$  diverges by the divergence test ( $\lim_{k \rightarrow \infty} (-1)^k \sqrt{k} \neq 0$ .)

$x=3$ :  $\sum_{k=1}^{\infty} \sqrt{k}$  diverges by the divergence test ( $\lim_{k \rightarrow \infty} \sqrt{k} \neq 0$ ).

So the interval of convergence is  $(1, 3)$

c. Let  $g(x)$  be an antiderivative for  $f(x)$ , i.e.  $\frac{dg}{dx} = f(x)$ , and such that  $g(2) = 0$ . What are the first four nonzero terms of the power series for  $g(x)$ ?

What are the radius and interval of convergence of the series for  $g(x)$ ?

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \cdot \frac{1}{(k+1)} \cdot (x-2)^{k+1} + C \quad (\text{integrating term by term})$$

Set  $x = 2$ :  $C = g(2) = 0$ .

$$\text{So } g(x) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \cdot \frac{(x-2)^{k+1}}{k+1}$$

The radius of convergence is the same  $R = 1$ .

Endpoints:  $x = 1$ :  $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}(k+1)}$  converges (alternating series test,  $\lim_{k \rightarrow \infty} \frac{1}{\sqrt{k}(k+1)} = 0$ , and  $\frac{1}{\sqrt{k}(k+1)} > \frac{1}{\sqrt{k+1}(k+1+1)}$ )

$x = 3$ :  $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k} \cdot (k+1)}$  converges by limit comparison

with the  $p$ -series  $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$   $\left( \lim_{k \rightarrow \infty} \left( \frac{1}{\sqrt{k}(k+1)} / \frac{1}{k^{3/2}} \right) = 1 \right)$

So the interval of convergence is  $[1, 3]$ .

7. (10 pts)

a. Give the first four nonzero terms of the Taylor series at  $x = 0$  for the function

$$f(x) = \frac{e^{x^2} + e^{-x^2}}{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

$$\frac{e^x + e^{-x^2}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} \frac{(1+(-1)^k)}{2} = 1 + \frac{x^4}{2!} + \frac{x^8}{4!} + \frac{x^{12}}{6!} + \dots$$

b. Find the Maclaurin series for  $x - \sin x$ . (Write out the first four nonzero terms.)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$x - \sin x = \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \dots$$

c. Using your answer to b, compute

$$\frac{x - \sin x}{x^3} = \frac{1}{3!} - \frac{x^2}{5!} + \frac{x^5}{7!} - \dots$$

by part (b)

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$$

So  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} = \frac{1}{3!} = \frac{1}{6}$  (the series on the right hand side evaluated at  $x=0$ ).