Solving for Constant Coefficient Linear Second Order Differential Equations
(in four easy steps)

Step 1: Write the equation in the form

$$\frac{d^2s}{dt^2} + b\frac{ds}{dt} + cs = 0,$$

where b and c are constants. (If you can't write it in this form, then you don't have a constant coefficient linear second order differential equation.)

Step 2: Write down the associated characteristic polynomial

$$x^2 + bx + c = 0,$$

where b and c are the same numbers as in Step 1, and find the roots of this polynomial

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Step 3: Look at the roots r_1 and r_2 that you got in Step 2, and pick the next sub-step accordingly:

Step 3a (Underdamped): If r_1 and r_2 are complex, then write $r_1, r_2 = \alpha \pm i\beta$, so that α and β are both real numbers. The general solution to the differential equation is

$$s = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t).$$

Step 3b (Overdamped): If r_1 and r_2 are both real and distinct (i.e., not the same), then the general solution is

$$s = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Step 3c (Critically Damped): If r_1 and r_2 are the same (i.e., a repeated root), then the general solution is

$$s = C_1 e^{r_1 t} + C_2 t e^{r_1 t}.$$

(Notice that since r_1 and r_2 are the same, it doesn't matter whether we use r_1 or r_2 in the formula in this subpart.)

Step 4:

- If you are looking for the general solution to the differential equation, you are done; your result in Step 3 is the final answer.
- If you are looking for a particular solution, however, then you must take the general solution that you obtained in Step 3 and plug in the initial values. For example, if you are told that $s(0) = 1$, then you take the general equation and substitute $s = 1$ and $t = 0$. Using the two initial conditions that you are given, you can find what the constants C_1 and C_2 are for your particular problem. (Don't forget that if the initial conditions involve s', then you must differentiate the general formula for s in order to find the general formula for s' in order to plug in the initial conditions.) Once you have determined the values of C_1 and C_2, you have your final answer (just put these particular values of C_1 and C_2 into your general solutions).

Graphing Your Solutions

Case a: Suppose your solution is

$$s = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t),$$

coming from Step 3a on the reverse side of this paper. In order to graph your solution, follow these steps:

a(i):

- Let $A = \sqrt{C_1^2 + C_2^2}$.
- Let ϕ be the angle satisfying $\sin\phi = \frac{C_1}{A}$, $\cos\phi = \frac{C_2}{A}$, and $\tan\phi = \frac{C_1}{C_2}$.
- Your solution is the same as

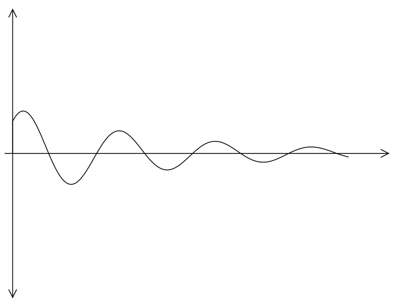
$$s = e^{\alpha t}\cdot A\sin(\beta t + \phi).$$

a(ii): Graph the function $A\sin(\beta t + \phi)$. (It is just a sine graph with amplitude A, period $2\pi/\beta$, and a shift to the left of ϕ/β.)

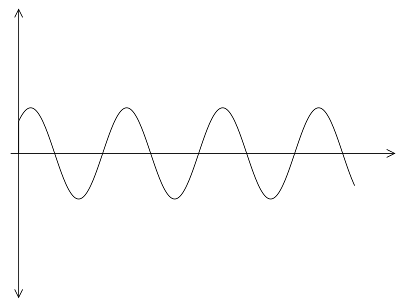
Example: If your function is $1.2\sin\left(3t + \frac{\pi}{4}\right)$, then it has amplitude 1.2, a period of $\frac{2\pi}{3}$, and is shifted to the left by $\frac{\pi}{12}$ units.

a(iii): Multiply the graph you got in part a(ii) by the factor $e^{\alpha t}$. If $\alpha = 0$, then nothing changes. If $\alpha > 0$, then the graph gets smaller and smaller. If $\alpha < 0$, then the graph gets bigger and bigger.

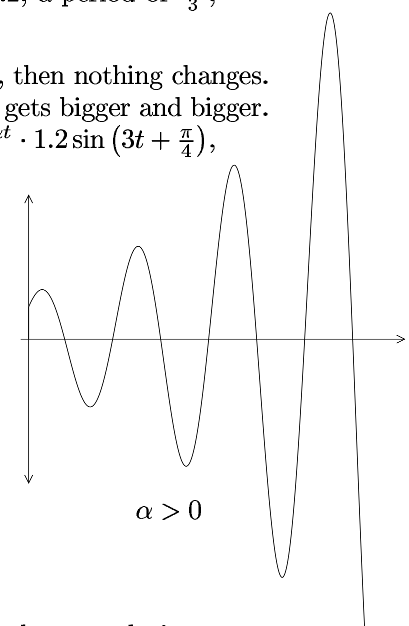
Example: Continuing the above example, the graphs below are for $e^{\alpha t}\cdot 1.2\sin\left(3t + \frac{\pi}{4}\right)$, for $\alpha = -0.3$, $\alpha = 0$, and $\alpha = 0.3$, respectively.



$\alpha < 0$



$\alpha = 0$



$\alpha > 0$

Case b: Suppose your solution is

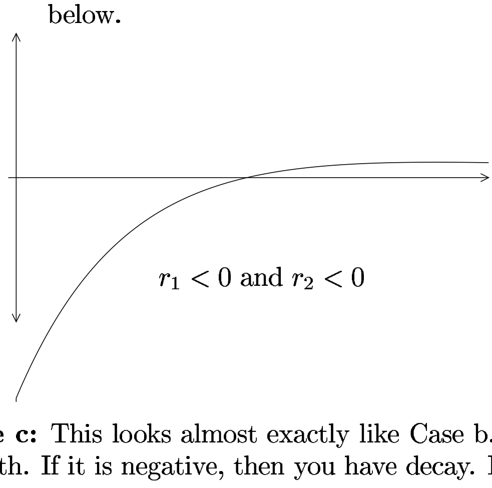
$$s = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$

coming from Step 3b on the reverse side of this paper. Follow these steps to graph your solution.

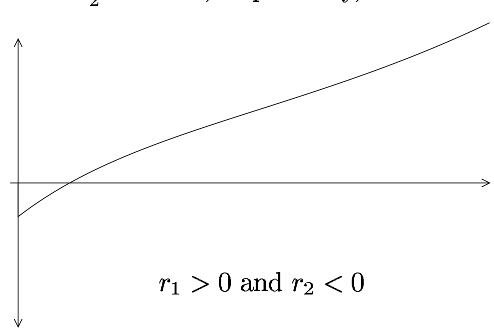
b(i): The function has either one or no zeros. If there is a zero, it will be $t = \frac{1}{r_1 - r_2}\ln\left(-\frac{C_2}{C_1}\right)$. (Warning: this formula only makes sense when there actually is one zero.)

b(ii): The graph of the function will either grow exponentially or decay exponentially. If both $r_1 < 0$ and $r_2 < 0$, then the function decays exponentially. If at least one of these two is positive, then it grows exponentially.

Example: The graphs of the functions $e^{-t} - 4e^{-3t}$ and $\frac{1}{2}e^{t} - e^{-3t}$, respectively, are shown below.



$r_1 < 0$ and $r_2 < 0$



$r_1 > 0$ and $r_2 < 0$

Case c: This looks almost exactly like Case b. If the single root r_1 is positive, then you have exponential growth. If it is negative, then you have decay. If there is a zero, then it is given by $t = -\frac{C_2}{C_1}$.