Last Review

Math 1B

Differential Equations!

A. Separation of Variables
B. Integrating Factors
C. Power Series Solutions
D. Checking Solutions
E. Concentration/Dilution Problems
F. Qualitative Analysis - from functions
G. - from graphs
H. 2^{nd} Order Differential Equations
I. Euler's Method

A. Separation of Variables
- use if can separate equation into y's one side, x's on other
- simplest technique - use if possible

(Note don't forget to include an integrating constant!)

1) Solve $x - y^2 \cdot y' = 0$ with $y(0) = 2$

2) Solve $\frac{dy}{dx} = \frac{x+1}{y^4+1}$

(B) Integrating Factors

- Use for diff. eqn's of form $y' + P(x)\,y = Q(x)$

Steps: Find $I(x) = e^{\int P(x)\,dx}$

don't need to write this down: Multiplying both sides of diff. eqn. yields

$$I(x)\left[y' + P(x)y\right] = \frac{d}{dx}\left[I(x)\cdot y\right] = I(x)\cdot Q(x)$$

so $$I(x)\cdot y = \int I(x)\cdot Q(x)\,dx$$

or $$y = \frac{1}{I(x)}\int I(x)\cdot Q(x)\,dx$$

3) Solve $y' = 4xy - 3x$ with $y(0) = 1$

(C) Power Series Solutions

→ write $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

so $y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$

and if needed: $y'' = 2a_2 + 6a_3x + \cdots$

write down as many a_n's as you're asked to find

→ sub these series in to your diff. eq. and match up powers of x, then solve for the coefficients a_n.

Note! if something is multiplied by a constant or x, then just multiply through, e.g.
$$3xy = 3x\,(a_0 + a_1x + a_2x^2 + \cdots)$$
$$= 3a_0x + 3a_1x^2 + 3a_2x^3 + \cdots$$

4) Solve $y'' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 0$ using power series.
Find the coefficients a_n for $0 \leq n \leq 5$.

(D) Checking Solutions

- Easy enough, use given function, calculate its derivative (and second derivative if necessary).
- Then sub back into original diff. eq., check to see if it's always true

5) Is $y(x) = 2e^{-x} + xe^{-x}$ a solution of $y'' + 2y' + y = 0$?

6) Is $y(x) = 3$ a solution of $y'' + 2y' + y = x$?

(E) Concentration / Dilution Problems

→ Let $A(t)$ = amount of substance (salt, pollutant, chemical, etc.)

→ Write $\begin{pmatrix}\text{rate of change}\\ \text{of amount}\end{pmatrix} = \dfrac{dA}{dt} = \begin{pmatrix}\text{rate}\\ \text{in}\end{pmatrix} - \begin{pmatrix}\text{rate}\\ \text{out}\end{pmatrix}$

<u>Note</u>: (rate in) is often just a constant (i.e. 3 lbs / minute)

(rate out) is usually of form $\left(\dfrac{\text{volume going out per time}}{\text{total volume}}\right) \cdot A(t)$

7) A tank holds 100 gallons of water containing 1 lb. of salt. A solution containing 1 lb. of salt per gallon is poured in at a rate of 3 gallons/minute, and the well-stirred mixture flows out the bottom of the tank at the same rate. Find the amount of salt in the tank at time t.

Note if volume going in is different from volume going out then the "total volume" you need in the (rate out) part varies with time too!

8) What is different in #7 if only 1 gallon of mixture goes in, per minute, while 3 go out per minute?

(F) Qualitative Analysis

If you're given $\frac{dy}{dt} = F(y)$, a function of y alone

Then: → Equilibrium solutions = roots of $F(y)$

→ sign of $F(y)$ for different values of y = sign of $\frac{dy}{dt}$

⇒ solution increases for $F(y)$ positive

decrease ... negative

(→ sometimes useful (only check if asked for it though!!)

concavity ⇒ $\frac{d^2y}{dt^2} = \frac{d}{dt}\left(F(y)\right)$ (need chain rule)

and point of inflections occur where $\frac{d^2y}{dt^2}$ changes sign

9) The number of birds in my attic is governed by the equation $\frac{dB}{dt} = 12B(200 - B)$ $\left(\begin{matrix} t=0 = \text{Jan } 1, '00 \\ t \text{ in days} \end{matrix}\right)$

a) Find equilibrium solutions for $B(t)$, the population of birds at time t.

b) graph (sketch as best as possible) possible solution functions for a starting population of 100 birds, and for a starting population of 300 birds.

c) What happens to the number of birds in my attic over the long run?

(G) Qualitative Analysis of Diff Eqs using Graphs.

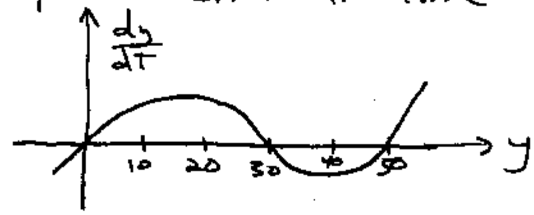
→ Same ideas as before.

① Know equil. solutions = places where graph of $\frac{dy}{dt} = 0$

② Solution functions increase where $\frac{dy}{dt}$ is positive. (decrease .. neg.)

③ pts of inflection at local max's / min's

10) The number of ants invading a picnic site at time t is given by solutions to the differential equation graphed at right.

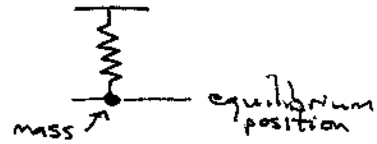


a) Graph possible solutions for $A(t)$ the number of ants at time t if the number of ants at time 0 is (respectively) 20, 30, 40 and 60.

b) What are the equilibrium solutions → are they stable or unstable?

(4) 2^{nd} Order Differential Equations

Basic example: springs –



If $s(t)$ gives the mass' position at time t, then from

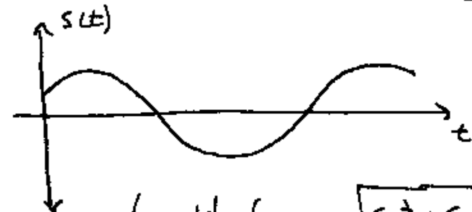
$F = m \cdot \underbrace{a}_{\frac{d^2s}{dt^2}}$ and $F = -\underbrace{k}_{\text{spring constant}} \cdot s$ (Hooke's Law)

then $\frac{d^2s}{dt^2} = -\left(\frac{k}{m}\right) \cdot s$ or $\boxed{\frac{d^2s}{dt^2} + \left(\frac{k}{m}\right)s = 0}$

is the differential equation for the motion of a spring (with no friction or "undamped")

general solution: $\boxed{s(t) = C_1 \cos\left(\sqrt{\frac{k}{m}}\, t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}\, t\right)}$

graphically these solutions look like simple oscillations (modified sine curves)



$$\left(\begin{array}{l} \text{amplitude} = \sqrt{C_1^2 + C_2^2} \\ \text{period} = \dfrac{2\pi}{\sqrt{k/m}} \end{array}\right)$$

Example problem

11) Solve the initial value problem $\frac{d^2s}{dt^2} = -2s$ with $s(0) = 10$ and $s'(0) = -20$.

(H) 2^{nd} Order Differential Equations continued

Given friction in a spring system then the differential equation governing the motion goes from

$$\frac{d^2s}{dt^2} + \left(\frac{k}{m}\right)s = 0 \quad \text{to} \quad \frac{d^2s}{dt^2} + \left(\frac{c}{m}\right)\frac{ds}{dt} + \left(\frac{k}{m}\right)s = 0$$

"undamped" — either "overdamped," "critically damped" or "underdamped"

Write this 2^{nd} order diff eq. as $\frac{d^2s}{dt^2} + b\frac{ds}{dt} + c\cdot s = 0$ — b, c constants

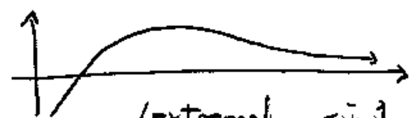
Then solve the $\boxed{\text{characteristic eqn}}$: $r^2 + br + c = 0$ for r and get 2 real roots, 1 real root or 2 complex conjugate roots

$\boxed{\text{Case 1}}$ "Overdamped" motion
→ lots of friction → no oscillations

mass settles back to equilibrium position

if $b^2 - 4c > 0$ → 2 real roots r_1, r_2
general solution is $\boxed{s(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$

$\boxed{\text{Case 2}}$ "Critically damped" motion
still too much friction for oscillations, but just



settles to equilibrium position

(extremely similar to overdamped case!)

if $b^2 - 4c = 0$ → one real root $-\frac{b}{2}$
general solution is $\boxed{s(t) = (C_1 t + C_2)\, e^{-bt/2}}$ where b is the coefficient for $\frac{ds}{dt}$

$\boxed{\text{Case 3}}$ "Underdamped" motion
some friction, but oscillations still occur just with smaller and smaller amplitudes

settles to equilibrium position over time

if $b^2 - 4c < 0$ → 2 complex conjugate roots $\alpha \pm \beta i$

general solution is
$$s(t) = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$
$$= e^{\alpha t}\left(C_1\cos(\beta t) + C_2\sin(\beta t)\right)$$

(in each case you should solve for the constants C_1 and C_2 if you're given an initial condition problem)

(H) 2nd order Differential Equations continued

12) Solve $y'' + 6y' + 8y = 0$

13) Solve $\frac{d^2s}{dt^2} + 2\frac{ds}{dt} + 2s = 0$ given $s(0) = 10$ and $s'(0) = -10$

IF This is the diff eqn. governing the motion of a mass on a spring, Then is the motion overdamped, critically damped, underdamped or undamped?

14) Give the general solution for $w'' + 6w' + 9w = 0$

(I) Euler's Method

Euler's method can be used to provide estimates for solutions to differential equations of the form $\frac{dy}{dx} = (\text{function of } x \text{ and } y)$, if you're given a specific starting or initial point. You have control over the accuracy of the estimates in your choice of size for Δx, the amount x changes by from step to step.

<u>Process</u>: (1) Make a table of values for x and y (start with the initial point given)

(2) Calculate $\frac{dy}{dx}$ at the initial point by subbing in the values of x and y at that point

(3) Having chosen, or been given, the step size Δx then find the x value for the next point in the table by adding Δx to the previous one

(4) Find the next value of y in the table by adding $\frac{dy}{dx} \cdot \Delta x$ to the previous y value ← (calculated in step (2))

(5) Repeat the process! Go back to (2), recalculate $\frac{dy}{dx}$ for the new x, y values in the table and keep going!

(15) Using a step size of 0.5, estimate $y(1.5)$ where $y(x)$ is the solution to $\frac{dy}{dx} = 2 - y$, starting from the initial point $y(0) = 1$

Answers

Differential Equations

Math 1B

1) $y^2 \cdot y' = y^2 \cdot \frac{dy}{dx} = x$ $\quad y^2\,dy = x\,dx$ $\quad \frac{y^3}{3} = \frac{x^2}{2} + c$ $\quad \to y^3 = \left(\frac{3}{2}x^2 + c'\right)$ $\quad y = \left(\frac{3}{2}x^2 + c'\right)^{1/3}$

and $2 = (0 + c')^{1/3} \Rightarrow c' = 8$

so $y = \left(\frac{3}{2}x^2 + 8\right)^{1/3}$

2) $(y^4+1)\,dy = (x+1)\,dx \Rightarrow \frac{y^5}{5} + y = \frac{x^2}{2} + x + c$
and can't solve explicitly for y

3) $y' - 4xy = -3x$
$I(x) = e^{\int -4x\,dx} = e^{-2x^2}$

then $e^{-2x^2} \cdot y = \int e^{-2x^2}(-3x)\,dx$. Let $u = -2x^2$
$du = -4x\,dx$
$-\frac{1}{4}du = x\,dx$

so $= \frac{3}{4}\int e^u\,du = \frac{3}{4}e^u + c$
$= \frac{3}{4}e^{-2x^2} + c$

so $y = \frac{1}{e^{-2x^2}}\left(\frac{3}{4}e^{-2x^2} + c\right) = \left(\frac{3}{4} + c\,e^{2x^2}\right)$

$y(0) = \left(\frac{3}{4} + c \cdot 1\right) = 1 \Rightarrow c = \frac{1}{4}$

so $y = \frac{3}{4} + \frac{1}{4}e^{2x^2}$

4) $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$
$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$
$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$

so $y'' + 4y = (2a_2 + 4a_0) + (6a_3 + 4a_1)x + (12a_4 + 4a_2)x^2 + (20a_5 + 4a_3)x^3 + \cdots$
$= 0$

now initial conditions imply $y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$
so we have $(2a_2 + 4a_0) = 2a_2 + 4 = 0$
$\Rightarrow a_2 = -2$

$(6a_3 + 4a_1) = 0 = 6a_3 + 0 \Rightarrow a_3 = 0$
$(12a_4 + 4a_2) = 0 = 12a_4 + 4(-2) \Rightarrow a_4 = \frac{2}{3}$
$(20a_5 + 4a_3) = 0 = 20a_5 \Rightarrow a_5 = 0$

4). continued

Thus $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$= 1 + 0\cdot x - 2x^2 + 0x^3 + \frac{2}{3}x^4 + 0\cdot x^5 + \cdots$

$= 1 - 2x^2 + \frac{2}{3}x^4 + \cdots$

5) $y(x) = 2e^{-x} + xe^{-x}$

$y'(x) = -2e^{-x} - xe^{-x} + e^{-x} = -e^{-x} - xe^{-x}$

$y''(x) = e^{-x} - (-xe^{-x} + e^{-x}) = e^{-x} + xe^{-x} - e^{-x} = xe^{-x}$

and then $y'' + 2y' + y = xe^{-x} + 2(-e^{-x} - xe^{-x}) + 2e^{-x} + xe^{-x}$

$= 0$ ✓

so yes it's a solution to $y'' + 2y' + y = 0$

6) However if $y(x) = 3$, then $y'(x) = 0$, $y''(x) = 0$,

and $y'' + 2y' + y = 0 + 2\cdot 0 + 3 = 3 \neq x$ for all x

so $y(x) = 3$ <u>is not</u> a solution of $y'' + 2y' + y = x$

7) $\frac{dA}{dt} = 3 - \left(\frac{3}{100}\right)A$ and $A(0) = 1$

where $A(t)$ = amount of salt in pounds at time t in minutes.

Then $\frac{dA}{3 - \left(\frac{3}{100}\right)A} = dt$, sub $u = 3 - \frac{3}{100}A$

$du = -\frac{3}{100}\,dA$, so $\int \frac{dA}{3 - \frac{3}{100}A}$

$\int \frac{dA}{3 - \frac{3}{100}A} = \int dt = t + c$

$= -\frac{100}{3}\int \frac{du}{u}$

$= -\frac{100}{3}\ln\left|3 - \frac{3}{100}A\right|$

so $-\frac{100}{3}\ln\left|3 - \frac{3}{100}A\right| = t + c$

$\ln\left|3 - \frac{3}{100}A\right| = -\frac{3t}{100} + c'$

$3 - \frac{3}{100}A = \pm e^{-\frac{3}{100}t + c'} = c'' e^{-\frac{3}{100}t}$

so $A(t) = 100 - c''' e^{-3/100\, t}$

now $A(0) = 1$, so $c''' = 99$

so $A(t) = 100 - 99e^{-3t/100}$

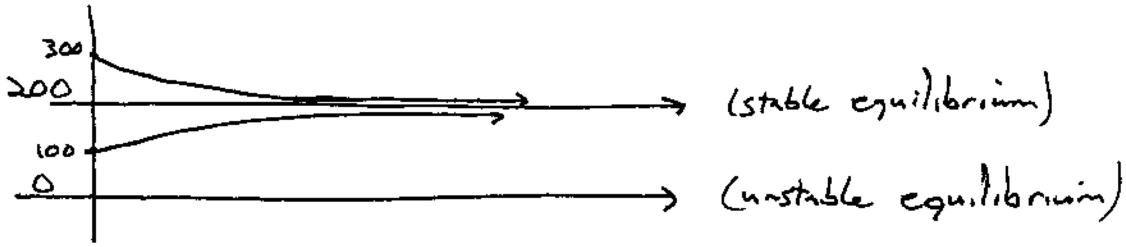
8) the differential equation now becomes

$$\frac{dA}{dt} = 1 - \left(\frac{3}{100-2t}\right) A$$

because total volume in tank goes down from 100 gallons to 100 − 2t after t minutes (1 gallon in 3 gallons out per minute)

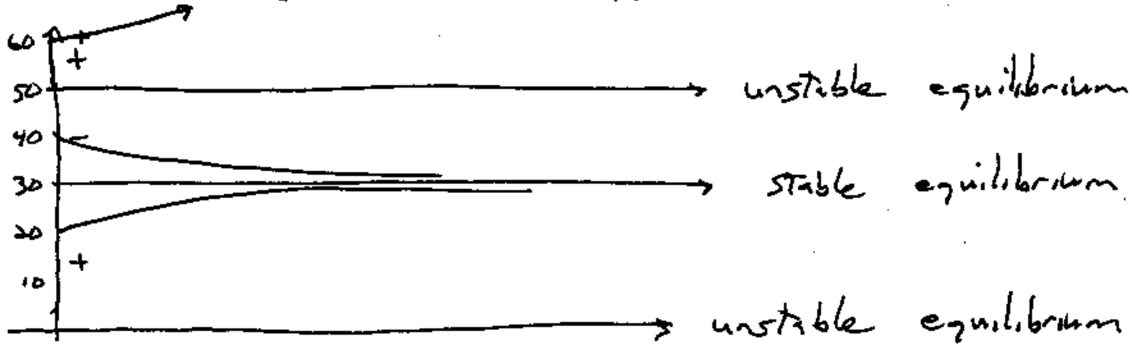
9) a) equilibrium solutions are at roots of $12B(200-B)$, so at $B = 0, 200$

b)



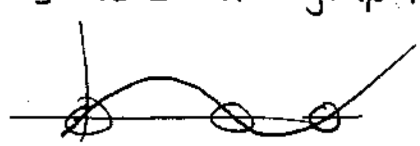
c) over long run, no matter how many birds you start with you end up approaching a population of 200 birds in the attic

10)



equilibrium solutions are Ant population $A(t) = 0, 30$ and 50

note there are three equilibriums, = 3 roots of graph



11) From $\frac{d^2s}{dt^2}+2s=0$ we know $s(t)=C_1\cos(\sqrt{2}\,t)+C_2\sin(\sqrt{2}\,t)$

solve for C_1, C_2: $\underset{=10}{s(0)}=C_1\cos(\sqrt{2}\cdot 0)+C_2\sin(\sqrt{2}\cdot 0)=\underset{=C_1}{C_1\cdot 1+C_2\cdot 0}$

so $C_1=10$

$\underset{=-20}{s'(0)}=-C_1\sin(\sqrt{2}\cdot 0)\cdot\sqrt{2}+C_2\cos(\sqrt{2}\cdot 0)\cdot\sqrt{2}=\sqrt{2}\cdot C_2$

so $C_2=-\frac{20}{\sqrt{2}}=-10\sqrt{2}$

so the solution is $s(t)=10\cos(\sqrt{2}\,t)-10\sqrt{2}\sin(\sqrt{2}\,t)$

12) characteristic equation is $r^2+6r+8=0$

here $6^2-4\cdot 2>0$ factors as $(r+4)(r+2)=0$

so 2 real roots so 2 real roots $-4, -2$

($\rightarrow$ overdamped if this equation was for a spring's motion)

general solution is then $y(t)=C_1e^{-2t}+C_2e^{-4t}$

(since we're not given initial conditions we cannot specify C_1 and C_2)

13) characteristic equation is $r^2+2r+2=0$

$2^2-4\cdot 2=-4<0$ so must be <u>underdamped</u> motion

roots are $\frac{-2\pm\sqrt{4-4\cdot 2}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm 2i}{2}=-1\pm i$ (or $\alpha=-1$, $\beta=1$)

so general solution is $s(t)=e^{-t}\left(C_1\cos(t)+C_2\sin(t)\right)$

since $s(0)=10$, and $s(0)=e^0\left(C_1\cos(0)+C_2\sin(0)\right)=C_1$

then $C_1=10$

$s'(t)=e^{-t}\left(-C_1\sin(t)\right)+C_1\cos(t)\cdot\left(-e^{-t}\right)+e^{-t}\left(C_2\cos(t)\right)+C_2\sin(t)\left(-e^{-t}\right)$

so $s'(0)=-e^0\cdot C_1+e^0\cdot C_2=-C_1+C_2=-10+C_2$,

since $s'(0)=-10=-10+C_2 \Rightarrow C_2=0$

so particular solution is $s(t)=e^{-t}\cdot 10\cos(t)$
$=10e^{-t}\cos(t)$

14) here $6^2-4\cdot 9=0$ so the solution is just

$w(t)=(C_1t+C_2)\,e^{-6t/2}=(C_1t+C_2)\,e^{-3t}$

Answers continued

(15) Make a table

x	y
0	1

initial point → (0, 1)
$y(0) = 1$

(2) calculate $\frac{dy}{dx} = 2 - y = 2 - 1 = 1$ using the values for x, y from the initial point

(3) new x value is $0 + 0.5$

(4) new y value is old one 1, plus $\frac{dy}{dx} \cdot \Delta x = 1 \times .5 = .5$

so new y-value is $1 + .5 = 1.5$

Now table is

x	y
0	1
.5	1.5

→ here $\frac{dy}{dx} = 2 - y = 2 - 1.5 = .5$

next value of x is $.5 + .5 = 1$

" " y is $1.5 + \frac{dy}{dx} \cdot \Delta x = 1.5 + (.5)(.5) = 1.75$

Now table is

x	y
0	1
.5	1.5
1	1.75

→ next $\frac{dy}{dx} = 2 - y = 2 - 1.75 = .25$

next x value is $1 + .5 = 1.5$

next y value is $1.75 + \frac{dy}{dx} \cdot \Delta x = 1.75 + (.25) .5$

$= 1.75 + .125 = 1.875$

so complete table is

x	y
0	1
.5	1.5
1	1.75
1.5	1.875

and so the estimate for $y(1.5)$ is 1.875

(and so to get a more accurate estimate you'd have to use a smaller step size for Δx)