Solutions - Claims about Definite Integrals

a) **Claim:** $0 < \int_0^1 e^x dx < 3.$ **THIS IS A CORRECT STATEMENT.**

- **Using antiderivatives:** $\int_0^1 e^x dx = \left[e^x\right]_0^1 = e^1 - e^0 \approx 1.7.$

- **Using areas:** The definite integral represents the area under the curve $y = e^x$ between $x = 0$ and $x = 1$. This area is shaded in Figure 1 below. This area is clearly positive, as $y = e^x$ never dips below the x-axis.

Figure 1: Area for the first claim.

The area of the entire graphing window is (base) × (height) = 3. The shaded area in Figure 1 fits inside the graphing window, so it must be less than 3, hence the definite integral is less than three.

b) **Claim:** $\int_{-1}^1 \sin(x)dx = 0.$ **THIS IS A CORRECT STATEMENT.**

- **Using antiderivatives:** $\int_{-1}^1 \sin(x)dx = \left[-\cos(x)\right]_{-1}^1 = -\cos(1) + \cos(-1) = 0$ as cosine is an even function.

- **Using areas:** The value of the definite integral is related to the area between the graph of $y = \sin(x)$ and the x-axis. Area above the x-axis contributes positively to the definite integral, whereas area below the x-axis contributes negatively to the definite integral. As there is exactly the same amount of area above as below when we look at the graph of $y = \sin(x)$ between $x = -1$ and $x = 1$, the two contributions to the definite integral cancel each other out leaving zero.

- **Function properties:** The definite integral of any *odd* function over an interval that is symmetric about the origin will be zero.

c) **Claim:** $\int_{\pi/2}^{3\pi/2} \cos(x)dx > 0$. **THIS IS AN INCORRECT STATEMENT.**

- **Using antiderivatives:** $\int_{\pi/2}^{3\pi/2} \cos(x)dx = \left[\sin(x)\right]_{\pi/2}^{3\pi/2} = -1-1 = -2$.

- **Using areas:** The definite integral is related to the area between the graph of $y = \cos(x)$ and the x-axis. As the area is below the x-axis, it will contribute negatively to the definite integral, making the definite integral negative rather than positive.

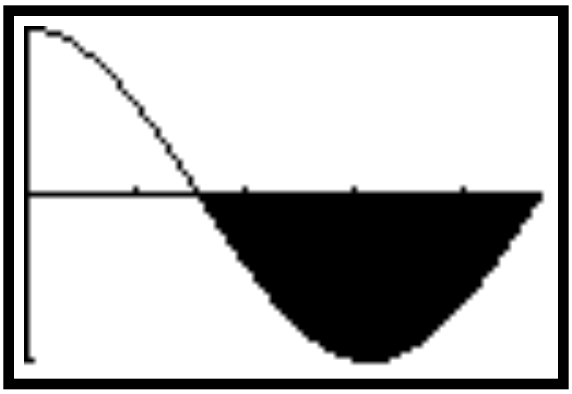

Figure 2: Area for the third claim.

d) **Claim:** $\int_0^1 e^{x^2} dx = \frac{1}{2}\int_{-1}^1 e^{x^2} dx$. **THIS IS A CORRECT STATEMENT.**

- **Using areas:** The first definite integral is the area between $x = 0$ and $x = 1$. The second definite integral is the area between $x = -1$ and $x = 1$. Because the function is symmetric around the y-axis, the amount of area between $x = -1$ and $x = 0$ is exactly the same as the amount of area between $x = 0$ and $x = 1$ (see Figure 3).

Figure 3: Areas for the fourth claim.

Since the area on each side of the y-axis is the same, the area to the right of the y-axis (that is, from $x = 0$ to $x = 1$) is half of the total.

- **Function properties:** If you have an *even* function and an interval that is symmetric about the y-axis, then the definite integral over either half of the interval is half of the integral over the entire interval.

e) **Claim:** $\int_0^1 \sqrt{1-x^2}\,dx < \frac{1}{2}$. **THIS IS AN INCORRECT STATEMENT.**

- **Using areas 1:** The equation $y = \sqrt{1-x^2}$ can be re-arranged to give: $x^2 + y^2 = 1$ which is the equation of a circle with radius 1, centered at (0, 0). The definite integral represents the area of one quarter of this circle (see Figure 4).

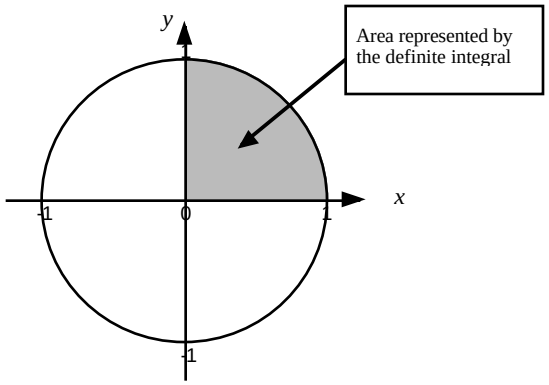

Figure 4: Comparison of area represented by definite integral with area of circle.

The value of the definite integral is:

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{1}{4}\pi\left(1^2\right) = \frac{\pi}{4} \approx \frac{3}{4}.$$

Since 3/4 is greater than 1/2, the claim is false.

- **Using areas 2:** (This argument is a little bit trickier.) Consider the area of the shaded triangle in Figure 5. The area of a triangle is one half of the base times the height. So, the area of this triangle is 1/2.

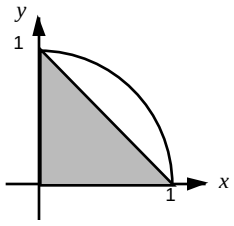

Figure 5: Shaded triangle that fits inside the area representing the definite integral.

As the triangle fits inside the quarter circle that represents the definite integral, the area of the circle must be less than the value of the definite integral. Therefore, one half is less than the value of the definite integral.