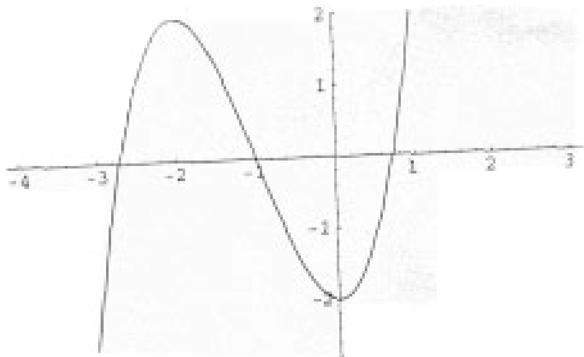
First Examination

Mathematics 1b

Fall, 2001

1. (12 points)

   Below is a graph of a twice differentiable function $f(x)$



   (a) The second degree Taylor polynomial for f at $x = 0$ is of the form $c_0 + c_1x + c_2x^2$. Determine the signs (positive, negative, or zero) of c_0, c_1, and c_2. Explain your reasoning briefly.

   (b) The second degree Taylor polynomial for f at $x = -2.2$ is of the form $c_0 + c_1(x + 2.2) + c_2(x + 2.2)^2$. Determine the signs of c_0, c_1, and c_2. Explain your reasoning briefly.

2. (16 points) For each of the following infinite series, determine if it converges or diverges. You must mathematically justify your answer in order to receive credit.

   (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{e}{5}\right)^{(n-1)}$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+1}{3n^2+n}$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{4^n\,n!}$

   (d) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{3}}{n(n+1)}$

3. (7 points) An air filtration system in a semiconductor factory passes air from outside through 8 filters, one after the other. Each filter removes 3/4 of the dust and other pollutants from the air that passes through it. If all the air sucked into the factory in one day contains 40 grams of dust, how much dust do the 8 filters together absorb? (You can express your answer as the sum of 8 terms.)

   What would be the total amount of dust absorbed if there were 50 such filters? Your answer to this part should involve the sum or difference of no more than a few terms. (This means that your answer shouldn't be expressed as the sum of 50 terms.)

4. (13 points)

   (a) Find the Taylor series centered about $x = 0$ for the function $2xe^{-x^3}$. Either include a general term for your series or use summation notation – whichever you prefer.

   (It would be wise not to do this from scratch, but to begin with a MacLaurin series that you either know or that can be computed quickly.)

   (b) Write the first four non-zero terms of the series expansion of $\int_0^{0.1} 2xe^{-x^3}\,dx$.

   (c) How many (non-zero) terms of the series are needed in order to compute $\int_0^{0.1} 2xe^{-x^3}\,dx$ with an error of less than $\frac{1}{10^8}$? Explain your reasoning.

5. (15 points)

(a) Construct the Taylor series representation of the function $f(x) = \frac{1}{1-x}$ at $x = 10$.

(b) What is the radius of convergence of this power series?

(c) If you evaluate the power series from part (a) at $x = 17$ will the series converge to a sum of $\frac{1}{1-17} = \frac{1}{-16}$? Explain very briefly.

(d) If you evaluate the power series from part (a) at $x = \frac{1}{2}$ will the series converge to a sum of $\frac{1}{1-\frac{1}{2}} = 2$? Explain very briefly.

6. (16 points) Suppose that $\sum_{n=0}^{\infty} a_n$ converges, and that $a_0, a_1, \ldots a_n, \ldots$ are all positive. Which of the following series *must* converge? Which of them *must* diverge? You must explain your reasoning clearly and carefully in order to receive credit. If it cannot be determined whether the series converges or diverges given the information, indicate this. An explanation is not needed in this case.

(i) $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ (ii) $\sum_{n=0}^{\infty} \frac{(-1)^n}{a_n}$ (iii) $\sum_{n=0}^{\infty} (\sin n) \cdot a_n$ (iv) $\sum_{n=0}^{\infty} \frac{5^n a_n}{4^n}$

7. (15 points) On a homework assignment you used the integral test to investigate the convergence of p–series. You may use the results from your homework assignment in this problem on alternating p–series. (You do not need to reuse the integral test.) For any real number p, the alternating p–series is the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} +$$

(a) Show that the alternating p–series diverges for $p \leq 0$.

(b) Show that the alternating p–series is convergent but not absolutely convergent if $0 < p \leq 1$

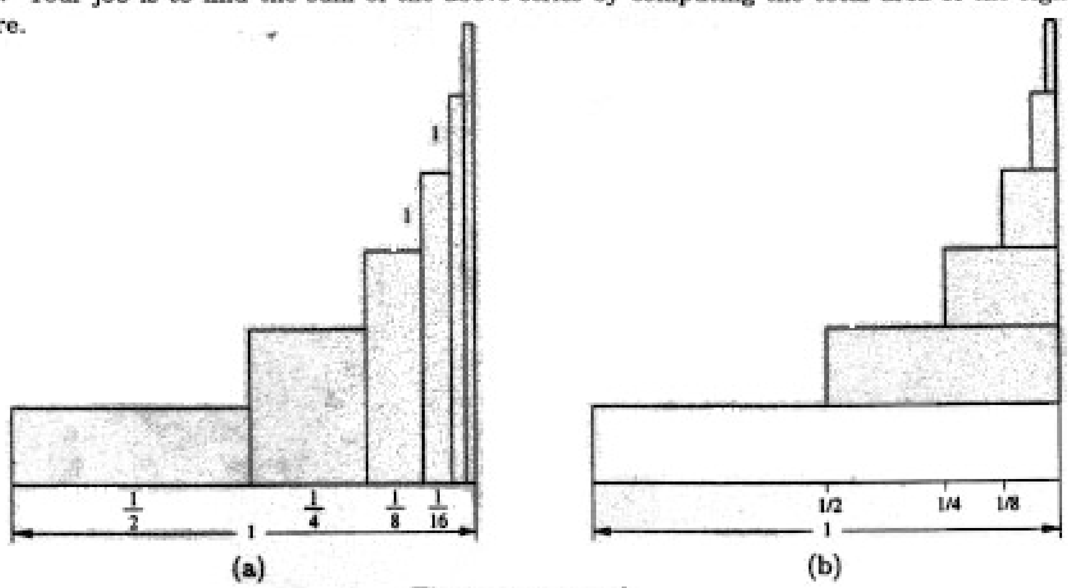
(c) Show that the alternating p–series is absolutely convergent for $p > 1$.

8. (6 points) Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

(a) Show that this series converges.

(b) The left hand figure below represents the sum of the above series. The right hand figure is identical except that instead of horizontal rectangles we draw vertical rectangles. This allows us to find the sum. Your job is to find the sum of the above series by computing the total area of the right hand figure.



Figures not to scale