

problems

1 Improper Integrals

1. For what values of p does $\int_1^\infty \frac{dx}{x^p}$ converge?
2. For what values of p does $\int_0^1 \frac{dx}{x^p}$ converge?
3. For what values of p does $\int_0^\infty \frac{dx}{x^p}$ converge?
4. Does the integral $\int_0^{10} \frac{dx}{x^2-3}$ converge?
5. Does the integral $\int_0^\infty e^{-x^3} dx$ converge?
6. (*) How could we obtain an approximate value for $\int_0^\infty e^{-x^3} dx$?
7. (*) Does $\int_1^\infty \frac{\sin x}{x^2}$ converge?
8. (*) Does $\int_1^\infty \frac{\sin x}{x}$ converge?

2 Integration by Slicing

9. Consider the region in the plane bounded by the parabola $x = y^2$ and the line $x = 9$. Now consider a space region with the given region as its base, and its cross-sections parallel to the y -axis are circles with diameter on the base region. What is the volume of the space region?
10. Derive the formula $V = \frac{4}{3}\pi R^3$ for the volume of a sphere using the method of slicing.
11. A conical frustum is the region formed by starting with a cone and then chopping it off by a plane parallel to the base. Find a formula for the volume of a conical frustum whose inner radius is r , outer radius is R , and height (the distance between the two bases) is h .

3 Volumes of Revolution

12. Explain how any given volume of revolution can be done using either washers or shells, *or* integrating with respect to x or with respect to y . Why might we prefer one method over the other?
13. Find the volume of the space region obtained by revolving the plane region enclosed by $x = y^2$ and $x = 2y + 3$ around the x -axis.

14. Use the method of washers to find the region obtained by taking the plane region enclosed by $y = x$, $y = \sqrt{x}$ about the line $y = 1$. When you finish, find the same problem using the method of shells, and check that your answers agree.

4 Arc Length

15. Find the arclength of the function $f(x) = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$. (You may use that $\int (\sec x) = \ln(\sec x + \tan x)$.)

5 Average Values

16. Let $f(x) = mx + b$ be a linear function on the interval $[c, d]$. Show that the average value of f is attained at the midpoint: $f((c+d)/2) = \frac{\int_c^d (mx+b) dx}{d-c}$.

6 Work problems

17. Filled with the desire to be alone, you have climbed a tree which is 10 meters high. So that you will be able to stay there for a while, you have arranged a basket containing lunch and some reading materials, weighing 15 kg in all. Before climbing up the tree you have attached a sturdy chain to the basket – the chain weighs 1 kg per meter. If you now pull the picnic basket up to you via the chain, how much work do you do?
18. Consider a spherical tank with radius 10 meters which is completely filled with water. Calculate the work done in pumping out the water through the top of the tank. (The mass density of water is 1000 kilograms per cubic meter, and the acceleration due to gravity is 9.8 meters per square second.)
19. Look back at the previous problem in terms of average values. What is the average distance that the water must be pumped? (Don't use calculus – exploit symmetry.) What is the total weight of the water in the tank? Why is the total work equal to the product of these two quantities?

7 Numerical Methods

20. Consider the function $f(x) = e^{x^2}$. i) Show that $f(x)$ is concave up on $(-\infty, \infty)$. ii) Use the midpoint tangent and trapezoidal rules to get bounds in both directions for $\int_0^1 e^{x^2} dx$, i.e. find numbers m and M such that $m < \int_0^1 e^{x^2} dx < M$.

8 Probability

21. All butterflies of the species *Fluturis Mathematicae* develop an intense blue coloring of their wings at some moment during their life. The probability density function for the occurrence of the blue color in time (where time is measured in days) is

$$f(t) = \begin{cases} 0, & \text{for } t \leq 0 \\ N \frac{e^{-3\sqrt{t}}}{\sqrt{t}}, & \text{for } t > 0 \end{cases} ,$$

where N is a constant to be determined.

- (i) Sketch the graph of $f(t)$.
- (ii) Find N so that $f(t)$ is indeed a probability density function.

- (iii) Find the probability that a butterfly's wings will become blue during the first two days of its life. Find the probability that the wings will turn blue between the 20th and 30th days of the butterfly's ephemeral life.