

Some More Challenging Questions for Second Midterm Examination Mathematics 1b

Please note that these questions are in no way indicative of either the level of difficulty nor the content of the examination. In particular, note that they do not cover all of the topics covered in section.

1. For each of the following, circle **T** if the statement is true, or **F** if the statement is false.
 - T F** If the area of the region bounded by the graphs of f and g is 1, then the area of the region bounded by the graphs of $h(x) = f(x) + C$ and $k(x) = g(x) + C$ is also 1.
 - T F** If f is continuous on $[0, \infty)$ and $\lim_{x \rightarrow \infty} f(x) = 0$, then $\int_0^{\infty} f(x) dx$ converges.
 - T F** If f is continuous on $[0, \infty)$ and $\int_0^{\infty} f(x) dx$ converges, then $\lim_{x \rightarrow \infty} f(x) = 0$.
 - T F** If f' is continuous on $[0, \infty)$ and $\lim_{x \rightarrow \infty} f(x) = 0$, then $\int_0^{\infty} f'(x) = -f(0)$.
 - T F** If the graph of f is symmetric with respect to the origin or the y -axis, then $\int_0^{\infty} f(x) dx$ converges if and only if $\int_{-\infty}^{\infty} f(x) dx$ converges.
 - T F** For any value of C , $y = (x - C)^3$ is a solution of the differential equation $y' = 3y^{\frac{2}{3}}$.
 - T F** If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution, where C is any real number.
 - T F** The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.
2. The area of the region bounded by the graphs of $y = x^3$ and $y = x$ *cannot* be found by the single integral

$$\int_{-1}^1 (x^3 - x) dx.$$

Explain why this is so. Use symmetry to write a single integral that does represent the area.

3. Let $f(t)$ be a function defined for all positive values for t . The Laplace Transform of $f(t)$ is defined by

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

if the improper integral exists. Find the Laplace Transform of the function $f(t) = t$.

4. The base of a solid is bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid for the following cross sections (taken perpendicular to the y -axis).
- square
 - semicircles
 - equilateral triangles
 - trapezoids for which $h = b_1 = \frac{1}{2}b_2$, where b_1 and b_2 are the lengths of the upper and lower bases
5. Prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.
6. Let a sphere of radius r be cut by a plane, thus forming a segment of height h . Show that the volume of this segment is $\frac{1}{3}\pi h^2(3r - h)$.
7. The graphs of $f(x) = x$ and $g(x) = ax^2$ intersect at the points $(0, 0)$ and $(1/a, 1/a)$. Find a so that the area of the region bounded by the graphs of these two functions is $\frac{2}{3}$.
8. Compute the average value of the function $f(x) = \sin(nx)$, on the interval $0 \leq x \leq \frac{\pi}{n}$, where n is a positive integer.
9. Let $y = f(x)$ be positive and strictly increasing on the interval $0 < a \leq x \leq b$. Consider the region R bounded by the graphs of $y = f(x)$, the x -axis, $x = a$, and $x = b$. If R revolved about the y -axis, show that the Disc Method and Shell Method yield the same volume.
10. Prove *Cavalieri's Theorem*: If two solids have equal altitudes and all plane sections parallel to their bases and at equal distances from their bases have equal areas, then the solids have the same volume.
11. The *Gamma Function* $\Gamma(n)$ is defined by
- $$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx, n > 0.$$
- Find $\Gamma(1)$, $\Gamma(2)$, and $\Gamma(3)$.
 - Use integration by parts to show that $\Gamma(n + 1) = n\Gamma$.
 - Express $\Gamma(n)$ in terms of factorial notation where n is a positive integer.
12. Suppose $\frac{dy}{dx} = -\frac{x}{y}$.
- Sketch several solution curves for the differential equation on a direction field.
 - If possible, identify the solutions. Then verify your guesses by differentiation.