Solutions to the Second Exam: Math 1b spring 2002

1. To do this problem you could look at the sign of y' and match it with that indicated in the graphs.

   (a) H

   (b) C

   (c) L

   (d) I

   (e) B

   (f) K

2. To find the equilibrium solutions, see if there is a constant value of y such that $y' = 0$. You can use your answers to problem 1 to determine stability if you like.

   (a) None.

   (b) The equilibrium solution $y = 0$ is stable.

   (c) The equilibrium solution $y = 1$ is unstable.

   (d) The equilibrium solution $y = 1$ is stable. The equilibrium solution $y = -2$ is unstable.

   (e) The equilibrium solution $y = 1$ is unstable. The equilibrium solution $y = -2$ is stable.

   (f) None.

3. (a) A (Look at the sign of the derivative.)

   (b) The differential equation
   $$\frac{dy}{dx} = \frac{-1}{2yx^2}$$
   can be solved by the technique of separation of variables. We have
   $$2y\,dy = \frac{-dx}{x^2}.$$
   Integrating both sides gives us $y^2 = 1/x + C$. Thus $y = \pm\sqrt{1/x + C}$. The initial condition gives us the unique solution
   $$y = \sqrt{1/x + 8/9}.$$

4. (a)
   $$\frac{dN}{dT} = \frac{N^2}{10000} - 0.01N.$$

   (b) The right hand side of this equation is
   $$(1/10000)N(N - 100),$$

   Therefore dN/dt is positive when $N > 100$, is zero when $N = 100$, and is negative when $0 < N < 100$. Therefore, if $N(0) = 10$, $N(t)$ will decrease (and asymptotically approach 0); if $N(0) = 100$, we have an equilibrium so that $N(t)$ will be always 100; if $N(0) = 1000$, $N(t)$ will increase.

5. (a) Since $\frac{lnx}{x}$ is undefined at $x = 0$, the integral $\int_0^1 \frac{\ln x}{x}\,dx$ is defined to be
   $$\int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\,dx$$
   with t approaching 0 from the right.

   To integrate, we can use u-substitution with $u = \ln x$. Then $du = \frac{1}{x}\,dx$ and the integral becomes
   $$\int \frac{\ln x}{x}\,dx = \int u du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

(Note that the limits of integration change from $x = t, x = 1$ to $u = \ln t, u = \ln 1 = 0$.)
This gives us

$$\begin{aligned}
\int_0^1 \frac{\ln x}{x}\,dx &= \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{x}\,dx \\
&= \lim_{t\to 0^+} \left[\frac{(\ln x)^2}{2}\right]_t^1 \\
&= \lim_{t\to 0^+} \left(\frac{(\ln 1)^2}{2} - \frac{(\ln t)^2}{2}\right)
\end{aligned}$$

But since $\lim_{t\to 0^+} \ln t = -\infty$, the limit does not exist. Therefore, the integral is DIVERGENT.

(b)

$$\begin{aligned}
\int_0^\infty e^{-x}\,dx &= \lim_{t\to\infty} \int_0^t e^{-x}\,dx \\
&= \lim_{t\to\infty} \left[-e^{-x}\right]_0^t \\
&= \lim_{t\to\infty} \left[-e^{-t} - (-e^{-0})\right] \\
&= 0 - (-1) = 1
\end{aligned}$$

because $\lim_{t\to\infty} e^{-t} = 0$.
Therefore, this integral is CONVERGENT, and its value is 1.

6. For part (a) we shall slice along the $x-$axis to get hollow cylindrical shells when we rotate. We first split the region R into 2 regions (R_1 and R_2) at the intersection of the function $y = 4x$ and the function $y = \frac{1}{x}$. We then slice each one region vertically into rectangles. The rotation of the rectangles around y axis will generate cylindrical shells. In general the i^{th} shell have height $y = f(x_i)$ circumference $2\pi x_i$ and thickness Δx, hence the volume of the i^{th} shell is $2\pi x_i f(x_i)\Delta x$ and the Riemann sum is

$$\sum_{i=i}^{n} f(x_i) 2\pi x_i \Delta x$$

Take the limit this converges to the integral:

$$\int_0^{\frac{1}{2}} 2\pi x(4x)dx + \int_{\frac{1}{2}}^1 2\pi x \frac{1}{x} dx$$

Which simplifies to

$$\int_0^{\frac{1}{2}} 8\pi x^2 dx + \int_{\frac{1}{2}}^1 2\pi dx$$

For part (b) we shall slice along the $x-$axis to get washers when we rotate. We again divide the region R into the same 2 regions. We again slice R vertically into rectangles and the volume generated by rotating the rectangles around the $x-axis$ are circular disks. The i^{th} disk has volume $\pi f(x_i)^2$. We again construct the Riemann sum and the integral is:

$$\int_0^{\frac{1}{2}} \pi(4x)^2 dx + \int_{\frac{1}{2}}^1 \pi \frac{1}{x^2} dx$$

7. Since the concentration of dioxin varies with the depth beneath the surface, we divide the pool up into thin horizontal slices on which the concentration of dioxin will be roughly constant. The amount of dioxin in a slice at depth h below the surface is given by

$$\Delta A = C(h)\Delta V = (3 + \frac{7}{100}h^2)\pi r^2 \Delta h$$

To find r, we use the Pythagorean theorem on the triangle shown in the figure. (This figure is on a separate page at the end of the solutions.) Since the pool is a hemisphere, the length of the hypotenuse is the radius of the sphere, or 10 meters. (Note that the part of the pool which is filled with water is *not* a hemisphere.) Thus

$$r^2 = 10^2 - (5+h)^2 = 100 - (5+h)^2.$$

The possible values for h run from $h = 0$ meters at the surface of the water to $h = 5$ meters at the very bottom of the pool Adding up the contributions of all the slices, we see that the total amount of dioxin is given by

$$\int_0^5 \pi(3 + \frac{7}{100}h^2)(100 - (5+h)^2)dh$$

nanograms.

8. Let $w(t)$ be the weight of the water and bucket at time t. Since water leaks at the rate of 0.3 pounds per second, $w(t) = 20 - 0.3t$, where $t = 0$ is the time at which the traveler starts to pull the bucket out of the well. We can also express w as a function of y, where y is the distance of the bucket of the surface of the water. Since the traveler lifts the bucket at 1.5 feet per second, $y = 1.5t$, so $w(y) = 20 - \frac{0.3}{1.5}y = 20 - 0.2y$. At the top of the well, $y = 30$, so the amount of water remaining is $w(30) = 20 - .2(30) = 14$ pounds.

   To compute the work done in lifting the bucket, we divide the process of lifting the bucket into many small steps in which the bucket is lifted from a height y to a height $y + \Delta y$. The basic formula is $\Delta W = Fd$. During each step, the force is approximately constant and is given by $w(y)$. (Note that since we are using English units there is no need to multiply by g.) The distance traveled in each step is just $y + \Delta y - y = \Delta y$. Thus $\Delta W = (20 - 0.2y)\Delta y$. The possible values of y range from $y = 0$ at the bottom of the well to $y = 30$ at the top. Thus the total work done is

$$\int_0^{30} (20 - 0.2y)dy$$

   foot-pounds.

   Alternately, we could divide the process of lifting the bucket into many small steps of duration Δt. During each step the weight of the bucket would be roughly constant and would be given by $w(t)$. The distance the bucket moves in a time Δt is $1.5\Delta t$, so $\Delta W = Fd = (20 - 0.3t)(1.5\Delta t)$. The time interval in which the traveler is lifting the bucket runs from $t = 0$ when he first begins to lift to $t = 30/1.5 = 20$ when the bucket reaches the top of the well. Thus the total work done is

$$\int_0^{20} 1.5(20 - 0.3t)dt$$

   foot-pounds.

9. (a) Since f is a probability density function, we must have
   $\int_{-\infty}^{\infty} f(t)\, dt = 1$.
   Since $f(t) = 0$ for $t < 0$, we can rewrite the integral as $\int_0^{\infty} f(t)\, dt$,
   which is an improper integral that must be interpreted as $\lim_{a\to\infty} \int_0^a f(t)\, dt$.
   We have
   $\int_0^a f(t)\, dt = \displaystyle\int_0^a \frac{C}{(2+t)^2}\, dt = -\frac{C}{(2+t}\,|_0^a = -\frac{C}{(2+a)} + \frac{C}{2+0}$,
   which (as $a \to \infty$) goes to $\frac{C}{2}$.
   Hence $\frac{C}{2}$, and $C = 2$.

   (b) To solve $\int_0^T f(t)\, dt = 0.75$, we write the integral as

$$-\frac{C}{(2+t)}\,|_0^T = -\frac{C}{2+T} + \frac{C}{2+0} = -\frac{2}{(2+T)} + 1$$

   and get

$$1 - \frac{2}{2+T} = 3/4,$$

or $T = 6$.
That is, the probability of having to wait no more than 6 minutes is exactly 75 percent.