Solutions to the First Examination: Fall 2001

1. a) The second degree Taylor polynomial for f at $x = 0$ is the quadratic given by

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2$$

- c_0 is negative because $c_0 = f(0)$ and $f(0)$, the value of f at 0, or the $y-$intercept, is negative.
- c_1 is zero because $c_1 = f'(0)$ and $f'(0)$, the slope of f at 0, is zero. (f has a local minimum at 0 and is differentiable at 0.)
- c_2 is positive because $c_2 = \frac{f''(0)}{2}$ and $f''(0)$, is positive. (f is concave up at 0.)

b) The second degree Taylor polynomial for f at $x = -2.2$ is the quadratic given by

$$f(-2.2) + f'(-2.2)(x + 2.2) + \frac{f''(2.2)}{2}(x + 2.2)^2$$

- c_0 is positive because $c_0 = f(-2.2)$ and $f(-2.2)$, the value of f at -2.2, is positive.
- c_1 is positive because $c_1 = f'(-2.2)$ and $f'(-2.2)$, the slope of f at -2.2, is positive. (f is increasing at -2.2)
- c_2 is negative because $c_2 = \frac{f''(-2.2)}{2}$ and $f''(-2.2)$, is negative. (f is concave down at -2.2.)

2. (a) This is a geometric series with ratio $r = \frac{e}{5}$. Since $e < 3$, hence $r < 1$ and the series converges.

(b) We will use the nth term test:

$$\begin{aligned} lim_{n\to\infty} a_n &= lim_{n\to\infty} \frac{2n^2+1}{3n^2+n} \\ &= lim_{n\to\infty} \frac{\frac{2n^2}{n^2}+\frac{1}{n^2}}{\frac{3n^2}{n^2}+\frac{n}{n^2}} \\ &= lim_{n\to\infty} \frac{2+\frac{1}{n^2}}{3+\frac{1}{n}} \\ &= \frac{2}{3} \end{aligned}$$

Since $lim_{n\to\infty} a_n \neq 0$ the series must diverge.

(c) We will use the ratio test:

$$\begin{aligned} lim_{n\to\infty} \mid \frac{a_{n+1}}{a_n} \mid &= lim_{n\to\infty} \frac{3^{n+1}}{4^{n+1}\cdot n!} \cdot \frac{4^n \cdot n!}{3^n} \\ &= lim_{n\to\infty} \frac{3}{4(n+1)} \\ &= 0 \end{aligned}$$

Since the limit of the ratio is less than one, the series converges absolutely.
Note we can also prove convergence by comparison test (compare with $\sum \frac{3^n}{4^n}$ or $\sum \frac{1}{n!}$). A third way of proving convergence is to realize this series as the MacLaurin series of $e^x = \sum \frac{x^n}{n!}$ with $x = \frac{3}{4}$. Since the MacLaurin series converges for all x, hence it must converge for $\frac{3}{4}$.

(d) We will use comparison test here. Since $n^2 < n(n+1)$ it follows that $\frac{1}{n^2} > \frac{1}{n(n+1)}$ and we have:

$$\sum \frac{\sqrt{3}}{n(n+1)} < \sum \frac{\sqrt{3}}{n^2}$$

$\sum \frac{1}{n^2}$ converges because it's a p series with p greater than 1. Hence $\sum \frac{\sqrt{3}}{n^2}$ converges and applying the comparison above, the series $\sum \frac{\sqrt{3}}{n(n+1)}$ converges.

3. We are given that each filter removes 3/4 of the dust that it encounters. Thus, the first filter absorbs $30 = (3/4) \cdot 40$ grams of dust and passes along $10 = (1/4) \cdot 40$ grams of dust. The second filter must then deal with the **10** grams of dust remaining. It absorbs $30/4 = (3/4) \cdot 10 = (3/4) \cdot (1/4) \cdot 40$ grams of the dust and passes along $10/4 = (1/4) \cdot 10 = (1/4) \cdot (1/4) \cdot 40 = (1/4)^2 \cdot 40$ grams of the dust to the third filter. The third filter must then deal with **10/4** grams of dust. It absorbs $30/16 = (3/4) \cdot (10/4) = (3/4) \cdot (1/4)^2 \cdot 40$ grams of the dust and passes along $10/16 = (1/4) \cdot (10/4) = (1/4) \cdot (1/4)^2 \cdot 40 = (1/4)^3 \cdot 40$ grams of the dust to the fourth filter.

So, after passing through three filters, the air still contains $(1/4)^3 \cdot 40$ grams of the original dust. How much did the first three filters absorb? We can compute this in at least two ways. For example, we can add the amount that each filter absorbs:

$$(3/4) \cdot 40 + (3/4) \cdot (1/4) \cdot 40 + (3/4) \cdot (1/4)^2 \cdot 40$$

grams of dust absorbed by the three filters. Or, we can just say that if the air started with 40 grams of dust and only $(1/4)^3 \cdot 40$ grams of dust remain after passing through three filters, then the filters must have absorbed the rest; *i.e.*, the filters must have absorbed

$$40 - (1/4)^3 \cdot 40$$

grams of dust.

Continuing this investigation, we see, using our first line of reasoning, that eight filters will absorb

$$\sum_{i=0}^{7} (3/4) \cdot (1/4)^i \cdot 40$$

grams of dust. Or, using our second line of reasoning

$$40 - (1/4)^8 \cdot 40$$

grams of dust will have been absorbed by the eight filters.

Now, after 50 filters, how much dust will the filters have absorbed?

By our first line of reasoning, the answer is

$$\sum_{i=0}^{49} (3/4) \cdot (1/4)^i \cdot 40$$

We recall the closed form for a geometric sum:

$$\sum_{i=0}^{n} a \cdot r^i = \frac{a \cdot (1 - r^{n+1})}{1 - r}$$

if $r \neq 1$. In our case a is $30 = 3/4 \cdot 40$, $r = 1/4$, and n is 49. Thus, the amount of dust absorbed will be

$$\frac{30 \cdot (1 - (1/4)^{50})}{1 - (1/4)}$$

grams.

By our second line of reasoning, the amount of dust absorbed by 50 filters will be

$$40 - (1/4)^{50} \cdot 40$$

grams.

You should verify that the two answers are the same.

4. (a) Find the Taylor series centered about $x = 0$ for the function $2xe^{-x^2}$.

   The Taylor series about 0 of e^x is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Replacing x by $-x^3$ gives the Taylor series $e^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$. Then multiplying by $2x$ gives $2xe^{-x^3} = \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{3n+1}}{n!}$.

(b) Write the first four non-zero terms of the series expansion of $\int_0^{0.1} 2xe^{-x^3}\, dx$.
By (a) we have

$$\int_0^{0.1} 2xe^{-x^3}\, dx = \int_0^{0.1} \left(\frac{2x}{0!} - \frac{2x^4}{1!} + \frac{2x^7}{2!} - \frac{2x^{10}}{3!} + - \cdots\right) dx = \left[x^2 - \frac{2x^5}{5} + \frac{x^8}{8} - \frac{x^{11}}{33} + - \cdots\right]_{x=0}^{0.1} = 10^{-2} - \frac{2}{5}10^{-5} +$$

(c) How many nonzero terms of the series are needed to compute the integral in (b) with an error of less than 10^{-6}?
The integral is expressed in (b) as an alternating series: the terms change signs, decrease in absolute value, and tend to 0. Therefore we may use the alternating series error estimate: the error after a partial sum is bounded by the absolute value of the first omitted term. Since the third term is $10^{-8}/8$, which is less than 10^{-6}, the first two terms are sufficient.

5. (a) To find the Taylor series of $f(x)$ at $x = 10$ one can compute the following:

$$f(10) + f'(10)(x-10) + \frac{f''(10)}{2!}(x-10)^2 + \frac{f'''(10)}{3!}(x-10)^3 + \ldots + \frac{f^{(n)}(10)}{n!}(x-10)^n + \ldots$$

$f(x) = \frac{1}{1-x}$	$f(10) = \frac{1}{(-9)}$
$f'(x) = (1-x)^{-2}$	$f'(10) = \frac{1}{(-9)^2}$
$f''(x) = 2(1-x)^{-3}$	$f''(10) = 2!\frac{1}{(-9)^3}$
$f'''(x) = (3)(2)(1-x)^{-4}$	$f'''(10) = 3!\frac{1}{(-9)^4}$
$f^{(4)}(x) = 4!(1-x)^{-5}$	$f(4)(10) = 4!\frac{1}{(-9)^5}$
$\ldots$	
$f^n(x) = n!(1-x)^{-(n+1)}$	$f^{(n)}(10) = n!\frac{1}{(-9)^{(n+1)}}$

This gives us

$$-\frac{1}{9} + \frac{1}{9^2}(x-10) - \frac{2!}{9^3 2!}(x-10)^2 + \frac{3!}{9^4 3!}(x-10)^3 + \ldots + (-1)^{n+1}\frac{n!}{9^{(n+1)}n!}(x-10)^n + \ldots$$

or

$$-\frac{1}{9} + \frac{(x-10)}{9^2} - \frac{(x-10)^2}{9^3} + \frac{(x-10)^3}{9^4} + \ldots + (-1)^{n+1}\frac{(x-10)^n}{9^{(n+1)}} + \ldots$$

An alternative method is the following: To find the Taylor series of $f(x)$ at $x = a$ is the same as to find the Maclaurin series of $f(a+t)$ at $t = 0$ and then plug $x = t - a$. In our case, $f(x) = \frac{1}{1-x}$ and $a = 10$ so that

$$\begin{aligned} f(10+t) = 1/(1-(10+t)) &= -1/(t+9) = (-1/9)(1/(1+t/9)) \\ &= -1/9(1 - t/9 + (t/9)^2 - (t/9)^3 + \cdots) \\ &= \sum_{n=0}^{\infty}(-1)^{n+1}t^n/9^{n+1}. \end{aligned}$$

Here $t = x - 10$.

(b) The geometric series $-1/9\sum_{n=0}^{\infty}(-1)^n\frac{(x-10)^n}{9^n}$ (with $a = -1/9$ and $r = \frac{(x-10)}{-9}$ converges to $\frac{-1/9}{1+\frac{x-10}{9}}$ for $|\frac{x-10}{-9}| < 1$ Therefore, for $|x - 10| < 9$ the series converges to $f(x)$. The radius of convergence is thus 9.

(c) Since $|17 - 10| < 9$, the series converges to $f(17)$.

(d) Since $|1/2 - 10| > 9$, the series diverges.

6. (*i*) The series $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ converges. This can be seen as follows. $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ are both series with positive terms, $\sum_{n=0}^{\infty} a_n$ converges, and for all $n \geq 0$, $\frac{a_n}{2^n} \leq a_n$. By the comparison test for convergence, we conclude that $\sum_{n=0}^{\infty} \frac{a_n}{2^n}$ also converges.

For this question, 1 point was gained by giving the correct answer and by providing a vague heuristic justification of the answer. 2 or 3 points rewarded partial or almost correct applications of the ratio test.

(*ii*) The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{a_n}$ diverges. Since $\sum_{n=0}^{\infty} a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$, therefore $\lim_{n\to\infty} \frac{1}{a_n} = \infty$, taking into account the fact that all a_ns are positive. Thus the sequence $\frac{(-1)^n}{a_n}$ diverges, and by the divergence test the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{a_n}$ diverges.

Here 1 or 2 points (depending on the rigor of expression) were given for mentioning that the terms of the series seem to increase. 3 points were given for writing '$\lim_{n\to\infty} \frac{1}{a_n} = \infty$'. Full credit was given for concluding that the series diverges by the nth term/divergence test. On this part many people tried to argue that the series *diverges* by noting that the conditions for the alternating series test to apply are not satisfied. This is logically wrong: the alternating series test can only be used to establish *convergence* of an alternating series. The fact that it fails to apply does not mean that the series in question diverges.

(*iii*) The series $\sum_{n=0}^{\infty} (\sin n) a_n$ converges, in fact it converges absolutely. Indeed, since $|\sin n| \leq 1$, we have that for all $n \geq 0$, $|\sin n| a_n \leq a_n$, and $\sum_{n=0}^{\infty} |\sin n| a_n$ converges by comparison with $\sum_{n=0}^{\infty} a_n$. This means that $\sum_{n=0}^{\infty} (\sin n) a_n$ also converges, because an absolutely convergent series is also convergent.

3 points were given here for concluding convergence of $\sum_{n=0}^{\infty} (\sin n) a_n$ upon comparison with $\sum_{n=0}^{\infty} a_n$. Full credit (4 points) was given for comparing the *absolute* series $\sum_{n=0}^{\infty} |\sin n| a_n$ with $\sum_{n=0}^{\infty} a_n$, the idea being that the comparison test for convergence only works for series with nonnegative terms.

(*iv*) It cannot be determined whether $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n a_n$ converges or diverges. If $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n$ ($a_n = \left(\frac{4}{5}\right)^n$) for instance, then $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n a_n = \sum_{n=0}^{\infty} 1$ is divergent. If $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$ ($a_n = \left(\frac{2}{5}\right)^n$), then $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n a_n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ is convergent.

0 points were given for declaring $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n a_n$ to be a divergent geometric series.

7) For any real number p, the alternating p-series is the infinite series

$$\Sigma_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \dots$$

a) Show that the alternating p-series diverges for $p \leq 0$.
Solution: There are two cases to consider.
$p = 0$: Then the general term is $(-1)^{n-1}$, so the limit of the nth term does not exist – the sequence alternates between 1 and -1. By the n-th term test, the series diverges.
$p < 0$: If p is negative, $\frac{1}{n^p} = n^{-p}$ and $-p$ is positive. The limit of a positive power of n as n approaches infinity is infinity, so the nth term goes to infinity in absolute value. Therefore the series again diverges by the nth term test.

b) Show that the alternating p-series is convergent but not absolutely convergent if $0 < p \leq 1$.
Solution: To see that the series is convergent, notice that 1) it is alternating, 2) the terms are decreasing: $n^p < (n+1)^p$ so $\frac{1}{n^p} > \frac{1}{(n+1)^p}$, and 3) the general term goes to zero: $n^p \to \infty$, so implies $\frac{1}{n^p} \to 0$. Therefore the alternating series test applies to show the convergence of the series. As for the lack of absolute convergence: when we take absolute values we get the conventional p-series, which we learned (using the integral test) diverges for $p \leq 1$.

c) Show that the alternating p-series is absolutely convergent for $p > 1$.

Solution: As above, absolute convergence means that the series must converge upon taking absolute values, but upon taking absolute values we again get the usual p-series. Since $p > 1$ it converges, which gives absolute convergence of the alternating p-series.

7. (a) Use the ratio test on the series.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{2} = \frac{1}{2} < 1$$

Therefore, the series converges, by the ratio test.

(b) The series corresponds to $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$. This is a geometric series that converges to $\frac{1}{1-\frac{1}{2}} = 2$.