Solutions For First Exam

1a. We can use the ratio test to check for convergence:

$$\lim_{m\to\infty} \frac{|x^{m+1}/(m+1)^2|}{|x^m/m^2|} = \lim_{m\to\infty} |x| \frac{m^2}{(m+1)^2} = |x|.$$

Thus, the series converges (absolutely) when $|x| < 1$ and diverges when $|x| > 1$. When $|x| = 1$ the test is inconclusive, and we must check those cases by hand. When $x = 1$, the series is a p-series with $p = 2 > 1$ and thus converges. When $x = -1$, it still converges absolutely (since the absolute values of its terms are the same as the terms when $x = 1$), and thus converges. (Alternatively, we can see that it converges when $x = -1$ using the alternating series test.)
Thus, the series converges for $|x| \le 1$, i.e., $-1 \le x \le 1$, and has radius of convergence 1.

1b. Consider the series

$$\sum_{m=1}^{\infty} \frac{(x-5)^m}{3^m m^2},$$

which is the same as

$$\sum_{m=1}^{\infty} \frac{\left(\frac{x-5}{3}\right)^m}{m^2}.$$

It results from replacing x with $(x-5)/3$ in the series from 1a, so it converges exactly for $|(x-5)/3| \le 1$. That is equivalent to $|x-5| \le 3$, i.e., $2 \le x \le 8$.

2a. We have

$$\sum_{j=1}^{n} jx^{j-1} = \frac{d}{dx}\sum_{j=1}^{n} x^j = \frac{d}{dx}\left(\frac{x - x^{n+1}}{1-x}\right) = \frac{(1-(n+1)x^n)(1-x) - (x - x^{n+1})(-1)}{(1-x)^2}.$$

Thus,

$$\sum_{j=1}^{n} jx^{j-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}.$$

2b. If we set $x = 1/2$ and take the limit as $n \to \infty$, then both $(n+1)x^n$ and nx^{n+1} tend to 0, and thus

$$\sum_{j=1}^{\infty} j\left(\frac{1}{2}\right)^{j-1} = \frac{1-0+0}{(1-1/2)^2} = 4.$$

2c. We want to compute

$$\lim_{x\to 1} \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2}.$$

Plugging in $x=1$ gives $0/0$, so we apply L'Hospitals rule:

$$\lim_{x\to 1} \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} = \lim_{x\to 1} \frac{-(n+1)nx^{n-1}+n(n+1)x^n}{-2(1-x)}.$$

Plugging in $x=1$ still gives $0/0$, so we apply L'Hospital's rule again, to get the limit

$$\lim_{x\to 1} \frac{-(n+1)n(n-1)x^{n-2}+n(n+1)nx^{n-1}}{2} = (n+1)n \lim_{x\to 1} \frac{-(n-1)x^{n-2}+nx^{n-1}}{2}.$$

Now we can set $x=1$ and simplify to get $n(n+1)/2$, and thus

$$\sum_{j=1}^{n} j = \lim_{x\to 1} \sum_{j=1}^{n} jx^{j-1} = \frac{n(n+1)}{2}.$$

3.

$$\sum_{a=36}^{\infty} 17\pi$$

diverges, since it is $17\pi + 17\pi + 17\pi + \cdots$. (Divergence test.)

$$\sum_{b=9}^{\infty} \frac{4^b}{3^{2b}} = \sum_{b=9}^{\infty} \left(\frac{4}{9}\right)^b$$

converges, since it is a geometric series with ratio $4/9 < 1$. (Be careful about simplifying the exponents; some people did this incorrectly.)

$$\sum_{c=32}^{\infty} \frac{1}{(c!)^{3/8}}$$

converges; we can use the ratio test:

$$\lim_{c\to\infty} \frac{1/(c+1)!^{3/8}}{1/(c!)^{3/8}} = \lim_{c\to\infty} \left(\frac{c!}{(c+1)!}\right)^{3/8} = \lim_{c\to\infty} \frac{1}{(c+1)^{3/8}} = 0 < 1.$$

Notice that this is **not** a p-series with $p = 3/8$.

$$\sum_{d=13}^{\infty} (\log d)^{-d}$$

converges, by the root test:

$$\lim_{d\to\infty} ((\log d)^{-d})^{1/d} = \lim_{d\to\infty} (\log d)^{-1} = 0.$$

$$\sum_{f=12}^{\infty} \frac{1}{\log f}$$

diverges, by comparison with the harmonic series (since $1/\log f > 1/f$).

$$\sum_{g=6}^{\infty} \frac{g^4+1}{g^6-1}$$

converges, by limit comparison with the sum of $1/g^2$:

$$\lim_{g\to\infty} \frac{(g^4+1)/(g^6-1)}{1/g^2} = 1,$$

and the sum of $1/g^2$ is a p-series with $p = 2 > 1$. (Be careful about manipulating exponents: the ratio $4/6$ does not enter anywhere into this problem.)

$$\sum_{h=3}^{\infty} \frac{(-1)^h}{h}$$

converges by the alternating series test.

$$\sum_{i=4}^{\infty} \frac{(-1)^{2i+1}}{\sqrt{i}}$$

diverges. At first glance, it might look like it alternates, but $2i+1$ is always odd, so $(-1)^{2i+1} = -1$ and all the terms are negative. Thus, it is -1 times a p-series with $p = 1/2 < 1$, so it diverges.

$$\sum_{j=88}^{\infty} 173589^{-5/j^2}$$

diverges, by the divergence test:

$$\lim_{j\to\infty} 173589^{-5/j^2} = 173589^0 = 1.$$

$$\sum_{k=5}^{\infty} \frac{(k-1)!}{k!}$$

diverges, since $(k-1)!/k! = 1/k$ so this is a p-series with $p = 1$ (the harmonic series, except we start with $k = 5$, which is irrelevant for convergence).

4a. We know that the Taylor series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

If we substitute in x^3 in place of x, we find that the 12-th degree Taylor polynomial for $e^{(x^3)}$ is

$$1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + \frac{x^{12}}{24}.$$

4b. One way to do this is by repeatedly differentiating. Set $f(x) = \log \sin x$. Then

$$f'(x) = \frac{\cos x}{\sin x} = \cot x,$$

$$f''(x) = -\csc^2 x,$$

$$f'''(x) = 2(\csc^2 x)(\cot x),$$

and

$$f^{(4)}(x) = -4(\csc^2 x)(\cot^2 x) - 2\csc^4 x.$$

Hence,

$$f(\pi/2) = 0,$$

$$f'(\pi/2) = 0,$$

$$f''(\pi/2) = -1,$$

$$f'''(\pi/2) = 0,$$

and

$$f^{(4)}(\pi/2) = -2.$$

It follows that the 4-th degree Taylor polynomial about $x = \pi/2$ is

$$-\frac{(x-\pi/2)^2}{2} - \frac{(x-\pi/2)^4}{12},$$

Another way to do this is by substitution. We have

$$\sin x = 1 - \frac{1}{2}(x-\pi/2)^2 + \frac{1}{24}(x-\pi/2)^4 + \cdots$$

(we can easily calculate all derivatives of $\sin x$ at $x = \pi/2$, or we can use $\sin x = \cos(x - \pi/2)$). We also know

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots.$$

Now we expand out

$$\log \sin x = \log\left(1 - \left(\frac{1}{2}(x-\pi/2)^2 - \frac{1}{24}(x-\pi/2)^4 + \cdots\right)\right).$$

We don't get many terms, because we care only about $(x - \pi/2)$ to the 4-th power or less. We have expand $\log \sin x$ as

$$-\left(\frac{1}{2}(x-\pi/2)^2 - \frac{1}{24}(x-\pi/2)^4 + \cdots\right) - \frac{1}{2}\left(\frac{1}{2}(x-\pi/2)^2 - \frac{1}{24}(x-\pi/2)^4 + \cdots\right)^2 + \cdots.$$

When we take only the terms involving $(x - \pi/2)$ to the 4-th power or less, we find that the 4-th degree Taylor polynomial is

$$-\frac{(x-\pi/2)^2}{2} - \frac{(x-\pi/2)^4}{12},$$

as before.
Either solution works fine. It depends on whether you prefer differentiating trigonometric functions, or substituting power series.

5a. We have

$$\sin 0.1 = 0.1 - \frac{0.1^3}{6} + \frac{0.1^5}{120} - \cdots.$$

This is an alternating series, so in the approximation

$$\sin 0.1 \approx 0.1 - \frac{0.1^3}{6},$$

the error is at most $0.1^5/120$, which is small enough to get four decimal places right. We have

$$0.1 - \frac{0.1^3}{6} = 0.1 - (0.001)(0.1666\ldots) = 0.099833333\ldots,$$

so to four decimal places,

$$\sin 0.1 \approx 0.0998.$$

5b. The error is at most

$$\frac{M|2-0|^6}{6!},$$

where M is an upper bound for $|f^{(6)}(x)| = |\sin(e^{2x})|$ between 0 and 2. Because the sine function is never larger than 1 in absolute value, $M = 1$ works. Thus, the error is at most $2^6/6!$, or $4/45$.

6a. We have

$$f(x) = (1-x^2)^{-1/3} = 1 + \sum_{k=1}^{\infty} \frac{(-1/3)(-1/3-1)\cdots(-1/3-(k-1))}{k!}(-x^2)^k,$$

by the binomial expansion. The first four non-zero terms are

$$1 + \frac{x^2}{3} + \frac{2x^4}{9} + \frac{14x^6}{81} + \cdots.$$

6b. The MacLaurin series for $g(x)$ is

$$4 - 3x + x^2 + \cdots$$

(as we can see from the derivatives we know at 0). Thus, we want to find the first three non-zero terms in

$$(4 - 3x + x^2 + \cdots)\left(1 + \frac{x^2}{3} + \frac{2x^4}{9} + \frac{14x^6}{81} + \cdots\right).$$

The constant term is $4 \cdot 1 = 4$. The coefficient of x is $-3 \cdot 1 = -3$. The coefficient of x^2 is $4 \cdot 1/3 + 1 \cdot 1 = 7/3$. We do not have enough information to determine the coefficient of x^3 (since we do not know the coefficient of x^3 in the MacLaurin series for $g(x)$), but we do not need to, since we only need the first three non-zero terms. Thus, the product is

$$4 - 3x + \frac{7}{3}x^2 + \cdots.$$

(Notice that we **cannot** compute the coefficients of the product just by multiplying corresponding coefficients in the two factors. If we are trying to find the coefficient of x^n in the product, we need to look at all the ways a term from each factor can multiply to give an x^n.)

7a. This series converges. Because the cosine is never larger than 1 in absolute value, we have

$$\left|\frac{2\cos(k^2)+1}{k^2}\right| \leq \frac{3}{k^2},$$

so the series converges absolutely by comparison with a p-series with $p = 2 > 1$.

7b. We have

$$\sum_{\ell=0}^{\infty} \frac{2^{\ell+2}3^{2\ell}}{5^{\ell-1}7^{\ell+1}} = \frac{2^2}{5^{-1}7}\sum_{\ell=0}^{\infty}\left(\frac{2\cdot 3^2}{5\cdot 7}\right)^{\ell} = \frac{20}{7}\sum_{\ell=0}^{\infty}\left(\frac{18}{35}\right)^{\ell} = \frac{20}{7}\frac{1}{1-18/35} = \frac{100}{17}.$$

7c. This is a telescoping series, which we can see most easily by writing it as

$$\left(\frac{\cos 1}{13e^1} - \frac{\cos 0}{13e^0}\right) + \left(\frac{\cos 2}{13e^2} - \frac{\cos 1}{13e^1}\right) + \left(\frac{\cos 3}{13e^3} - \frac{\cos 2}{13e^2}\right) + \cdots.$$

The k-th partial sum is

$$-\frac{\cos 0}{13e^0} + \frac{\cos k}{13e^k},$$

since everything else cancels. When we let $k \to \infty$, we find that the sum is

$$-\frac{\cos 0}{13e^0} = -\frac{1}{13}.$$