Additional Problems for Mathematics 1b
Handout D

1. In this problem you'll compare the error analysis arrived at using the alternating series error estimate with that gotten using the Taylor Remainder.

   Let $f(x) = x^{1/3}$.

   (a) Find the third order Taylor Polynomial, $T_3(x)$, for f at $x = 27$.

   (b) Use $T_3(x)$ found in part (a) to approximate $28^{1/3}$.

   (c) Find an upper bound for the error in this approximation by using the alternating series error estimate.

   (d) Now find an upper bound for the error in this approximation by using the Taylor Remainder (i.e., Taylor's Inequality).

2. The interval of convergence of the Maclaurin series for $\ln(1+u)$ is $u \in (-1, 1]$. On this interval the series converges to $\ln(1+u)$.

   (a) Using any method you like, find the Maclaurin series for $\ln(1+u)$.

   (b) By setting $u = x - 1$ in part (a), find a power series expansion for $\ln x$ centered at $x = 1$.

   (c) Find the Taylor series for $\ln x$ at $x = 1$ by taking derivatives. Make sure your answers to parts (b) and (c) agree. (They ought to because *if* a function has a power series expansion in $(x - 1)$ then that expansion will be the Taylor series about $x = 1$.)

3. Multiplying Series:

   (a) Find the third degree Taylor polynomial (centered at 0) for $f(x) = \sin x \cdot \cos x$ by multiplying Taylor series for $\sin x$ and $\cos x$ and stopping when you know that all remaining terms are of degree 4 or higher. Check your answer by using the identity $\sin 2x = 2 \sin x \cos x$.

   (b) In your last homework set you found the series for $\sqrt{1+x}$. (It was §8.8 #1: the answer is in the back of the book.) Multiply this series by itself to find the second degree Taylor polynomial for $\sqrt{1+x} \cdot \sqrt{1+x}$. Without computing, what do you expect the coefficient of the x^3 term to be?

4. Amanda asked her friend Charlie for some help when studying for a math test on series. Charlie had quick and easy methods, but he sometimes said things that are incorrect. Your job in this problem is to pay very careful attention to Amanda and Charlie's conversation and correct Charlie wherever necessary.

   - Amanda asked Charlie how you can tell if the series, $1 + (-1) + (-1)^2 + (-1)^3 + (-1)^4 + ...$ converges or diverges. Charlie answered, "That's easy. This is just a geometric series with $a = 1$ and $r = -1$. So you just plug into the formula, $\frac{a}{1-r} = \frac{1}{1-(-1)} = \frac{1}{2}$. So, it converges to one half." Do you agree with Charlie's statement? Explain why or why not.
   - Amanda told Charlie that she was having a lot of trouble with geometric series. She asked Charlie how you could find out whether an expression like: $3 + 3 \cdot 7^2 + 3 \cdot 7^4 + 3 \cdot 7^6 + ... + 3 \cdot 7^{20}$ converged or not, and if it did, what it converged to. Charlie answered, "Okay, these are two step problems. First, that's a geometric series with $a = 3$, $r = 7^2$. So, to get the total, you just plug into the formula: $\frac{a(1-r^{n+1})}{1-r} = \frac{3(1-49^{21})}{1-49}$."

     Charlie told Amanda that you can work that out on a calculator. (Is Charlie's closed form correct? Explain below.)

     "You can tell the series doesn't converge" Charlie continued," because the r is 49. That's greater than one, so the series doesn't converge." Is Charlie's statement about convergence accurate? Why or why not?

- The last question that Amanda asked was about a general infinite series like $\sum_{k=1}^{\infty} a_k$. She said that she remembered hearing something about looking at the value of a_k when k gets really, really big, and that can tell you something about whether the series $\sum_{k=1}^{\infty} a_k$ converges or not. Charlie responded, "Oh yeah, this makes convergence and divergence really easy. If you look at what the formula for a_k is, and if that formula gets closer and closer to zero then the series converges. Otherwise, the series diverges." Do you think Charlie is gave Amanda very solid advice? What advice would you have given her?

Note on §8.9 #25:
In #25 we are looking at the situation for which d is much much smaller than D, so $\frac{d}{D}$ is small. We would like to use only the first few terms of a Taylor series, so we need to write a series in u where u is very small. That's why the book suggests writing a series in powers of $u = \frac{d}{D}$. Begin by manipulating the expression given until you have

$$\frac{q}{D^2} - \frac{q}{D^2}\left[\frac{1}{\left(1+\frac{d}{D}\right)^2}\right] = \frac{q}{D^2}\left[1 - \frac{1}{\left(1+\frac{d}{D}\right)^2}\right].$$

You'll expand the expression $\frac{1}{\left(1+\frac{d}{D}\right)^2}$. This looks like $\frac{1}{(1+u)^2}$ where you can think of $\frac{d}{D}$ as u. You'll only use the first two terms of the series to get the desired result.