Handout M

1. Solve the following differential equations for $y(t)$. Give the general solution.

   a) $y'' + 6y' = 7y$

   b) $y'' + 6y' + 9y = 0$

   c) $y'' + 5y' + 6y = 0$

2. For each of the differential equations in the problem above, suppose that the initial conditions are $y(0) = -2$ and $y'(0) = 0$.

   (i) Use the initial conditions to find $y(t)$.

   (ii) Find $\lim_{t\to\infty} y(t)$.

3. Interpret $x(t)$ as the position of a mass on a spring at time t where $x(t)$ satisfies

$$x'' + 4x' + 3x = 0.$$

   Suppose the mass is pulled out, stretching the spring one unit from its equilibrium position, and given an initial velocity of $+2$ units per second.

   (a) Find the position of the mass at time t.

   (b) Determine whether or not the mass ever crosses the equilibrium position of $x = 0$.

   (c) When (at what time) is the mass furthest from its equilibrium position? Approximately how far from the equilibrium position does it get?

4. (a) Suppose that $x(t) = C_1e^{at} + C_2e^{bt}$. Show that $x(t) = 0$ at most once. Find the value of t for which $x(t) = 0$ if such a value exists.

   (b) Suppose that $x(t) = C_1e^{at} + C_2te^{at}$. Show that $x(t) = 0$ at most once. Find the value of t for which $x(t) = 0$ if such a value exists.

   (c) Conclude from parts (a) and (b) that if the characteristic equation of $x'' + bx' + cx = 0$ has either one real root or two real roots then the differential equation cannot model a mass at the end of a spring in the scenario that the mass oscillates back and forth around the equilibrium point.

5. Let's try to make sense of the expression $e^{(a+bi)t}$, that is, e raised to a complex number $a + bi$ where $i = \sqrt{-1}$. To do this, first observe that $e^{(a+bi)t} = e^{at} \cdot e^{bit}$, where a and b are real numbers. The part we must make sense of is e^{bit}. Use the Maclaurin Series for e^x to expand e^{bti}. Gather all terms with i and all terms without i. (Factor out the i from the terms with i.) Now rewrite e^{bit} in terms of familiar functions.

   Given your work above, what's $e^{\pi i}$?