Solutions to the First Math 1b Exam, Spring of 2002

1. With $f(x) = \sin x$, we have

   $f'(x) = \cos x$,
   $f''(x) = -\sin x$,
   $f'''(x) = -\cos x$,
   $f^{(4)}(x) = \sin x$, and
   $f^{(5)}(x) = \cos x$,

   so

   $f(\pi/4) = \sin \pi/4 = \sqrt{2}/2$,
   $f'(\pi/4) = \cos \pi/4 = \sqrt{2}/2$,
   $f''(\pi/4) = -\sqrt{2}/2$,
   $f'''(\pi/4) = -\sqrt{2}/2$,
   $f^{(4)}(\pi/4) = \sqrt{2}/2$, and
   $f^{(5)}(\pi/4) = \sqrt{2}/2$.

   So the 5th degree Taylor polynomial for $\sin x$ centered at $\pi/4$ is

$$
\begin{aligned}
T_5(x) &= \sum_{n=0}^{5} \frac{f^{(n)}(\pi/4)}{n!}(x-\pi/4)^n \\
&= \sqrt{2}/2 + (\sqrt{2}/2)(x-\pi/4) \\
&\quad -(\sqrt{2}/(2\cdot 2!))(x-\pi/4)^2 - (\sqrt{2}/(2\cdot 3!))(x-\pi/4)^3 \\
&\quad +(\sqrt{2}/(2\cdot 4!))(x-\pi/4)^4 + (\sqrt{2}/(2\cdot 5!))(x-\pi/4)^5.
\end{aligned}
$$

2. (a) **VI**
   (b) **I**
   (c) **V**
   (d) **IV**
   (e) **III**

   Explanation: The Taylor polynomial for a function may only approximate that function within a certain radius of its center. Therefore, we cannot assume that the Taylor polynomial is near the function at any point except the center, which in this case is $a = 2$. In order to match the polynomials with their generating functions, consider the values $f(x)$, $f'(x)$ and $f''(x)$. These will give us the value of the function, its slope and its curvature, all at the center $a = 2$. But these are also given in the coefficients of the Taylor polynomial:

$$T_2(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2$$

   When the first coefficient of the Taylor polynomial is positive, then so is the value $f(2)$. When the coefficient of $(x-2)$ in the Taylor polynomial is positive, then so is the slope $f'(2)$ of the function at 2. When the coeffiecient of the $(x-2)^2$ term is positive, so is the curvature $f''(2)$, and the function f must be concave up at 2. Likewise, a negative coefficient will indicate a negative value, slope or curvature, and a coefficient of zero will indicate a root, a local extremum, or an inflection point.

   As an example, the polynomial

$$1-(x-2)^2$$

has first coefficient $f(2) = 1$ so the value of the function at 2 must be postive. The second coefficient is $f'(2) = 0$, so the slope is zero and the function has a local extremum. The third coefficient $\frac{f''(2)}{2!} = -1$ so the curvature is negative, and f must be concave down at 2.

5 (a) We shall use the test for divergence (nth term test). As n goes to infinity, $ln(n)$ increase without bound, $lim_{n\to\infty}(-1)^n ln(n)$ does not exist. By the test for divergence the series divergence.

Note: This series fails the alternating series test also, but alternating series test can only show convergence. It is wrong to argue that the series diverges because it fails the alternating series test.

(b) We shall use the alternating series test. It is obviously alternating because of $(-1)^n$. Since $ln(n)$ is an increasing function, hence $\frac{1}{ln(n)}$ decreases. $lim_{n\to\infty}\frac{1}{ln(n)} = 0$ since $lim_{n\to\infty} ln(n) = \infty$. All 3 conditions of the alternating series are satisfied, hence the series converges.

(c) We shall use the comparison test. Since $sin^2(n) \leq 1$ for all n, we have $0 \leq \frac{sin^2(n)}{n^3} \leq \frac{1}{n^3}$. $\sum \frac{1}{n^3}$ is a p series with $p = 3 > 1$ and hence converges. Therefore the smaller series $\sum \frac{sin^2(n)}{n^3}$ must converge.

(d) There are two solutions.

Solution 1: we shall use the ratio test:

$$\begin{aligned} lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{6}{3^{n+1}+1} \cdot \frac{3^n+1}{6} \right| & (1) \\ &= \lim_{n\to\infty} \left| \frac{3^n+1}{3^{n+1}+1} \right| & (2) \\ &= \lim_{n\to\infty} \left| \frac{1+\frac{1}{3^n}}{3+\frac{1}{3^n}} \right| & (3) \\ &= \frac{1}{3} & (4) \end{aligned}$$

Solution 2: We shall use the comparison test. Since $3^n + 1 > 3^n$, we have $0 < \frac{1}{3^n+1} < \frac{1}{3^n}$ for all n. $\sum_n \frac{1}{3^n}$ is a geometric series with constant ratio $\frac{1}{3}$ and hence converges. The smaller series $\sum_n \frac{1}{3^n+1}$ must converge also.

6. (a) If 80% of the drug is removed every 8 hours, then 20% remains in the bloodstream. The ratio is $r = .2 = \frac{1}{5}$. The amount present after 7 doses will be

$$\underset{\substack{\uparrow \\ dose\ 7}}{400} + \underset{\substack{\uparrow \\ dose\ 6}}{\tfrac{1}{5}(400} + \underset{\substack{\uparrow \\ dose\ 5}}{\tfrac{1}{5}(400} + \underset{\substack{\uparrow \\ dose\ 4}}{\tfrac{1}{5}(400} + \underset{\substack{\uparrow \\ dose\ 3}}{\tfrac{1}{5}(400} + \underset{\substack{\uparrow \\ dose\ 2}}{\tfrac{1}{5}(400} + \underset{\substack{\uparrow \\ dose\ 1}}{\tfrac{1}{5}(400}))))))$$

$$\begin{aligned} &= 400 + \left(\frac{1}{5}\right)400 + \left(\frac{1}{5}\right)^2 400 + \left(\frac{1}{5}\right)^3 400 + \left(\frac{1}{5}\right)^4 400 + \left(\frac{1}{5}\right)^5 400 + \left(\frac{1}{5}\right)^6 400 \\ &= \sum_{n=0}^{6} 400\left(\frac{1}{5}\right)^n \quad or \quad \sum_{n=1}^{7} 400\left(\frac{1}{5}\right)^{n-1} \end{aligned}$$

(b) The exact (i.e. not approximate) answer in closed form is:

$$S_{42} = \frac{400 - 400\left(\frac{1}{5}\right)^{42}}{1 - \frac{1}{5}}$$

Because the sum $S_{42} = 400 + \left(\frac{1}{5}\right)400 + \cdots + \left(\frac{1}{5}\right)^{41}$, it makes sense that the exponent in the closed form is **42**.

(c) If dose a were taken every 8 hours, the total amount in the bloodstream would approach

$$S = \frac{a}{1-r} = \frac{a}{1-\frac{1}{5}} = \frac{a}{\left(\frac{4}{5}\right)}$$

Setting $S = 1000mg$, we get

$$\begin{aligned} 1000mg &= \frac{a}{\left(\frac{4}{5}\right)} \\ \left(\frac{4}{5}\right) 1000mg &= a \\ 800mg &= a \end{aligned}$$

7 (a) Find the Taylor series of $f(x)$ centered at $x = 0$. Hint: It would *not* be wise to start differentiating $f(x)$

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots & &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \\ \cos(2x^4) &= 1 - \frac{(2x^4)^2}{2!} + \frac{(2x^4)^4}{4!} - \dots & &= \sum_{n=0}^{\infty} \frac{(-1)^n (2x^4)^{2n}}{(2n)!} \\ &= 1 - \frac{4x^8}{2!} + \frac{16x^16}{4!} - \dots & &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{8n}}{(2n)!} \\ x^3 \cos(2x^4) &= x^3 - \frac{4x^11}{2!} + \frac{16x^{19}}{4!} - \dots & &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{8n+3}}{(2n)!} \\ x^3 \cos(2x^4) - x^3 &= -\frac{4x^11}{2!} + \frac{16x^{19}}{4!} - \dots & &= \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{8n+3}}{(2n)!} \end{aligned}$$

(b) What is $f^{(10)}(0)$? (That is, what is the tenth derivative of f evaluated at zero?)
To find $f^{(10)}(0)$, we can differentiate the series of part a) 10 times and evaluate at $x = 0$. Since the lowest order term in the Taylor series of $f(x)$ is a multiple of $x^1 1$, the lowest order term in the series for $f^{(10)}(x)$ will be a multiple of x. Thus $f^{(10)}(0) = 0$.
Alternately, we can recall that the Taylor series of $f(x)$ centered at 0 has the general form

$$\sum_{n=0} \infty \frac{f^{(n)}(0)}{n!} x^n$$

Since there is no term with an x^{10} in the series we found in part a), the coefficient of the x^{10} term in the Taylor series must be 0, so $f^{(10)}(0) = 0 \cdot 10! = 0$.

(c) What is $f^{(11)}(0)$?
As in b), we differentiate the Taylor series for $f(x)$ 11 times and evaluate at $x = 0$. The term $\frac{-4}{2!} x^1 1$ in the series for $f(x)$ will give us a constant term $\frac{-4}{2!} \cdot 11!$ in the series for $f^{(11)}(x)$, so $f^{(11}(0) = \frac{-4}{2!} \cdot 11! = (-2)11!$. This result can also be obtained by using the general formula for the 11th term in the Taylor series of $f(x)$.

8 (a) Use the Taylor series representation of e^{-x^2} to find $\int_0^{0.1} e^{-x^2}\,dx$.
We recall that e^x has associated MacLurian series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Thus, the MacLurian series for e^{-x^2} is

$$\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

So, since this series converges to e^{-x^2} for all real numbers x, we can use the result about integrating power series to obtain:

$$\begin{aligned}\int_0^{.1} e^{-x^2}\,dx &= \int_0^{.1} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \int_0^{.1} \frac{(-1)^n \cdot x^{2n}}{n!}\,dx \\ &= \sum_{n=0}^{\infty} \Big|_0^{.1} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)\cdot n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (.1)^{2n+1}}{(2n+1)\cdot n!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}\cdot(2n+1)\cdot n!}.\end{aligned}$$

(b) Approximate $\int_0^{0.1} e^{-x^2}\,dx$ with error less than 10^{-5}. You may leave your answer as a sum. Explain your reasoning.
Since the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}\cdot(2n+1)\cdot n!}$ is alternating and satisfies the conditions of the alternating series test (AKA Leibniz's theorem), the alternating series estimation theorem tells us that the difference between the actual sum of the series (*i.e.*, $\int_0^{.1} e^{-x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}\cdot(2n+1)\cdot n!}$) and the kth partial sum is given by the $(k+1)$st term. Writing down the first few terms, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{10^{2n+1}\cdot(2n+1)\cdot n!} = \frac{1}{10} - \frac{1}{3\cdot 10^3} + \frac{1}{10^5\cdot 5\cdot 2!} - \frac{1}{10^7\cdot 7\cdot 3!} + \cdots.$$

Since the third term in the right-hand sum is less than 10^{-5}, we see that $\frac{1}{10} - \frac{1}{3\cdot 10^3}$ approximates $\int_0^{.1} e^{-x^2}\,dx$ to within 10^{-5}.

9 We define a function $f(x)$ by setting

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{\sqrt{n}\;2^n}$$

for those x for which the series converges.

(a) Find the radius of convergence.
We use the ration test. We first write down the quotient of the $(n+1)$st term over the nth term:

$$\left|\frac{x^{2(n+1)}\cdot 2^n\cdot\sqrt{n}}{\sqrt{n+1}\cdot 2^{n+1}\cdot x^{2n}}\right| = \frac{x^2}{2}\cdot\sqrt{\frac{n}{n+1}}.$$

The limit of this quotient is $x^2/2$. So, by the ratio test the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\cdot x^{2n}}{\sqrt{n}\cdot 2^n}$$

converges for all those x which satisfy $x^2/2 < 1$. In other words, the series converges for all those x in the interval $(-\sqrt{2},\sqrt{2})$. So, the radius of convergence is $\sqrt{2}$.

(b) Use the definition of $f(x)$ to find a series that converges to $f'(1)$, that is, the derivative of f evaluated at $x = 1$.

For x in the interval $(-\sqrt{2}, \sqrt{2})$, we can interchange summation and differentiation. Therefore, for these x we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} \sum_{1}^{\infty} \frac{(-1)^n \cdot x^{2n}}{\sqrt{n} \cdot 2^n} \\ &= \sum_{1}^{\infty} \frac{d}{dx} \frac{(-1)^n \cdot x^{2n}}{\sqrt{n} \cdot 2^n} \\ &= \sum_{1}^{\infty} \frac{(-1)^n \cdot 2n \cdot x^{2n-1}}{\sqrt{n} \cdot 2^n} \end{aligned}$$

So, since 1 lives in the interval $(-\sqrt{2}, \sqrt{2})$, we have

$$f'(1) = \sum_{1}^{\infty} \frac{(-1)^n \cdot 2n}{\sqrt{n} \cdot 2^n} = \sum_{1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{2^{(n-1)}}.$$

(c) Write down the third partial sum for the series for $f'(1)$. Is f increasing or decreasing at 1?
The three term partial sum is $-1 + \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{4}$. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \sqrt{n}}{2^{(n-1)}}$$

is alternating and satisfies the conditions of the alternating series test (Leibniz's theorem). Therefore, we see that $f'(1) < 0$ and so f is decreasing at 1.