Section 5.6 #6, 18:

6. Let $u = \sin^{-1} x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then
$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \dfrac{x}{\sqrt{1-x^2}}\,dx$. Setting $t = 1 - x^2$, we get $dt = -2x\,dx$, so
$-\int \dfrac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\frac{1}{2}\,dt\right) = \frac{1}{2}\left(2t^{1/2}\right) + C = t^{1/2} + C = \sqrt{1-x^2} + C$. Hence,
$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$.

18. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \Rightarrow du = 2x\,dx$, $v = -e^{-x}$. By (6),
$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$. Next let
$U = x$, $dV = e^{-x}\,dx \Rightarrow dU = dx$, $V = -e^{-x}$. By (6) again,
$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So
$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1} + 1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

Section 5.7 #18, 20, 24, 27, 29:

18. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x - 1 = A(x+2) + B(x+1)$.
Substituting -2 for x gives $-3 = -B \Leftrightarrow B = 3$. Substituting -1 for x gives $-2 = A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{3}{x+2}\right)dx = \left[-2\ln|x+1| + 3\ln|x+2|\right]_0^1$$
$$= (-2\ln 2 + 3\ln 3) - (-2\ln 1 + 3\ln 2) = 3\ln 3 - 5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

20. $\dfrac{x^2+2x-1}{x^3-x} = \dfrac{3x^2-1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get
$x^2 + 2x - 1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$. Substituting 0 for x gives $-1 = -A \Leftrightarrow A = 1$.
Substituting -1 for x gives $-2 = 2B \Leftrightarrow B = -1$. Substituting 1 for x gives $2 = 2C \Leftrightarrow C = 1$. Thus,
$\int \dfrac{x^2+2x-1}{x^3-x}\,dx = \int \left(\dfrac{1}{x} - \dfrac{1}{x+1} + \dfrac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\dfrac{x(x-1)}{x+1}\right| + C$.

24. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get
$x^2 - x + 6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \Leftrightarrow A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \Rightarrow B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

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$$\begin{array}{r|l} & x \\ x^2+1 & x^3 \\ & \underline{x^3+x} \\ & -x \end{array}$$

By long division, $\dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$. Thus,

$$\int_0^1 \frac{x^3}{x^2+1}\,dx = \int_0^1 x\,dx - \int_0^1 \frac{x\,dx}{x^2+1}$$
$$= \left[\tfrac{1}{2}x^2\right]_0^1 - \tfrac{1}{2}\int_1^2 \frac{1}{u}\,du \quad [\text{where } u = x^2+1,\ du = 2x\,dx]$$
$$= \tfrac{1}{2} - \left[\tfrac{1}{2}\ln u\right]_1^2 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2 = \tfrac{1}{2}(1 - \ln 2)$$

29. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4 \left(1+\frac{4}{u^2-4}\right)du \text{ [by long division]}$$

$$= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)}.$$

Multiply $\frac{1}{(u+2)(u-2)} = \frac{A}{u+2} + \frac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A + B = 0$ and $-2A + 2B = 1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so $\frac{1}{(u+2)(u-2)} = \frac{-1/4}{u+2} + \frac{1/4}{u-2}$ and the last integral is

Handout E #1, 2:

1. Show that $\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}9\pi$ - in other words, show analytically that the area of a circle of radius 3 is 9π by doing the following:

   We'd like to eliminate $\sqrt{9-x^2}$ by making a substitution that makes the integrand a perfect square. We will exploit the trig identity $\sin^2 t + \cos^2 t = 1$, or, equivalently, $9\sin^2 t + 9\cos^2 t = 9$ We know that $9 - 9\sin^2 t$ is a perfect square, so we'll use the substitution $x = 3\sin t$. Now we need to write the entire integral in terms of t.

   a) If $x = 3\sin t$ then what is dx in terms of t?
      $dx = 3\cos(t)dt$
   b) If $x = 3\sin t$ then what is $\sqrt{9-x^2}$ in terms of t?
      $\sqrt{9-x^2} = \sqrt{9-9\sin^2 t} = \sqrt{9\cos^2 t} = 3\cos t$
   c) If $x = 3\sin t$ then what are the new endpoints of integration in terms of t?
      $0, \frac{\pi}{2}$
   d) Write the integral in terms of t.
      $\int_0^{\frac{\pi}{2}} 9\cos^2 t dt$
   e) Evaluate the integral in (d).
      $\int_0^{\frac{\pi}{2}} 9\cos^2 t dt = 9(\frac{t}{2} + \frac{\sin(2t)}{4}|_0^{\frac{\pi}{2}}) = 9 * \frac{\pi}{4} = \frac{1}{4}9\pi$
   f) Conclude that the area of a circle of radius 3 is 9π.
      Because t represents the angle, the area from $t = 0$ to $t = \frac{\pi}{2}$ is one quarter of a circle with radius 3. Thus, the full area of the circle is $4 * \frac{1}{4}9\pi = 9\pi$.

2. Find $\int \sin^2\theta\,d\theta$.
   $\int \sin^2\theta d\theta = \int \frac{1}{2}(1-\cos(2\theta))d\theta = \int \frac{1}{2}d\theta - \int \frac{1}{2}\cos(2\theta)d\theta = \frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + C$