Problem Set 6

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x,\ du = \cos x\,dx\}$ and $\{u = \cos x,\ du = -\sin x\,dx\}$, respectively.

1. $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx \overset{c}{=} \int (1-u^2)u^2(-du)$
$= \int (u^2-1)u^2\,du = \int (u^4-u^2)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

2. $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x\,dx \overset{s}{=} \int_0^1 (1-u^2)^2 du$
$= \int_0^1 (1-2u^2+u^4)du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1-\sin^2 x)\cos x\,dx$
$\overset{s}{=} \int_1^{\sqrt{2}/2} u^5(1-u^2)du = \int_1^{\sqrt{2}/2} (u^5-u^7)du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2}$
$= \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = -\frac{11}{384}$

8. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\int_0^{\pi/4} \tan^2 x \sec^4 x\,dx = \int_0^{\pi/4} \tan^2 x \sec^2 x(\sec^2 x\,dx) = \int_0^{\pi/4} \tan^2 x(1+\tan^2 x)(\sec^2 x\,dx)$$
$$= \int_0^1 u^2(1+u^2)du = \int_0^1 (u^2+u^4)\,du$$
$$= \left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$$

$\int \frac{x}{\sqrt{4+x}}\, dx$ $\quad u = 4+$ $\quad du \quad 2x\,dx$

$\frac{1}{2} \quad u \quad du \quad 2 \quad \frac{^{1/2}}{1/2}\Big]$ $\quad \boxed{\sqrt{4+x^2} + C}$

$\int x\sqrt{4-}\, dx \quad du \quad 2x\,dx \quad \frac{x}{2}\,dx \quad 4\, u\, du$

$\hookrightarrow \quad u^{1/2}\, 4\, u\, du \quad \frac{1}{2}\, 4u \quad u^{3/}\, du \quad 2\left[\frac{^{3/2}}{3/2} \quad \frac{^{5/2}}{5/2}\right]$

$\left[\frac{4}{3}(4 \quad ^{3/} \quad \frac{1}{5} \quad x^{5/2}\right] \quad \left[\left(\frac{4}{3} \quad ^{3/2} \quad \frac{1}{5} \quad ^{5/2}\right) \quad \left(\frac{4}{3} \quad \frac{1}{5}\cdot 32\right.\right.$

$\sim \boxed{0.45615489}$

$\int \frac{x}{\sqrt{9-x^2}}\, dx \quad du \quad 2x\,dx \quad \frac{x^2}{2}\,dx \quad \frac{9-}{} \quad (9- \quad du$

$\hookrightarrow \int \frac{9-}{u}\, du \quad \frac{1}{2} \quad ^{1/2}\, du \quad du)$

$\frac{1}{2}\left[\frac{9u^{1/2}}{1/2} \quad \frac{u^{3/2}}{3/2}\right] \quad \left[\quad ^{1/2} \quad \frac{1}{3}\, 9-x \quad ^{3/2}\right]$

$\left[\quad \frac{1}{3} \quad ^{1/2} \quad 3 \quad \frac{1}{3}\, 27\right] \quad \boxed{0.0866282099}$

4. a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$\int e^{-x^2}\,dx = \int 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \,dx = \boxed{x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n+1}}{(2n+1)\, n!}}$$

b) $\int_0^{.5} \sqrt{1+x^3}\,dx$

$$(1+x^3)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (x^3)^n = 1 + \frac{1}{2}x^3 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^6}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^9}{3!}$$
$$+ \cdots + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)\frac{x^{3n}}{n!} + \cdots$$

$$\int \sqrt{1+x^3}\,dx = x + \frac{1}{2}\frac{x^4}{4} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^7}{7\cdot 2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^{10}}{10\cdot 3!}$$
$$+ \cdots + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-n+1\right)\frac{x^{3n+1}}{(3n+1)\, n!} + \cdots$$

consider all but the first term as an alternating series:

the term $\left|\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^{10}}{10\cdot 3!}\right| \overset{?}{<} 0.0001$

$$\Rightarrow \frac{1}{160}\cdot|x^{10}| < 0.0001 \Rightarrow |x^{10}| < 0.0160 \Rightarrow |x| < .6613\ldots$$

both limits of integration fall in this range $(-.6613\ldots < x < .6613\ldots)$.

$$\Rightarrow \int_0^{.5} \sqrt{1+x^3}\,dx \approx \left[x + \frac{x^4}{8} - \frac{1}{56}x^7\right]_0^{.5}$$

$$\doteq 0.507672991 \approx \boxed{0.5077}$$