4. $25 = f(x) = kx = k(0.1)$ [10 cm = 0.1 m], so $k = 250$ N/m and $f(x) = 250x$. Now 5 cm = 0.05 m, so $W = \int_0^{0.05} 250x\,dx = \left[125x^2\right]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31$ J.

7. The portion of the rope from x ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted x_i^* ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} \tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50} \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

10. The work needed to lift the bucket itself is 4 lb · 80 ft = 320 ft-lb. At time t (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40 - 0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40 - 0.2\left(\frac{1}{2}x_i^*\right)\right]$ lb of water when it is x_i^* ft above its original 80 ft depth. Moving this amount of water a distance Δx requires $\left(40 - \frac{1}{10}x_i^*\right)\Delta x$ ft-lb of work. Thus, the work needed to lift the water is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\left(40 - \tfrac{1}{10}x_i^*\right)\Delta x = \int_0^{80}\left(40 - \tfrac{1}{10}x\right)dx = \left[40x - \tfrac{1}{20}x^2\right]_0^{80} = (3200 - 320) \text{ ft-lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.

12. A horizontal cylindrical slice of water Δx ft thick has a volume of $\pi r^2 h = \pi \cdot 12^2 \cdot \Delta x$ ft^3 and weighs about $(62.5 \text{ lb/ft}^3)(144\pi\,\Delta x \text{ ft}^3) = 9000\pi\,\Delta x$ lb. If the slice lies x_i^* ft below the edge of the pool (where $1 \le x_i^* \le 5$), then the work needed to pump it out is about $9000\pi x_i^*\,\Delta x$. Thus,

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 9000\pi x_i^*\,\Delta x = \int_1^5 9000\pi x\,dx = \left[4500\pi x^2\right]_1^5 = 4500\pi(25 - 1) = 108{,}000\pi \text{ ft-lb}$$

17. (a) $W = \displaystyle\int_a^b F(r)\,dr = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1m_2\left[\frac{-1}{r}\right]_a^b = Gm_1m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

(b) By part (a), $W = GMm\left(\dfrac{1}{R} - \dfrac{1}{R + 1{,}000{,}000}\right)$ where M = mass of earth in kg, R = radius of earth in m, and m = mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,

$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1000) \times \left(\dfrac{1}{6.37 \times 10^6} - \dfrac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9$ J.

18. (a) $W = \displaystyle\int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm\lim_{t\to\infty}\left(\frac{-1}{t} + \frac{1}{R}\right)$

$= \dfrac{GMm}{R}$, where M = mass of earth = 5.98×10^{24} kg, m = mass of satellite = 10^3 kg, R = radius of earth = 6.37×10^6 m, and G = gravitational constant = 6.67×10^{-11} N-m^2/kg^2. Therefore,

work $= \dfrac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^{10}$ J.

(b) From part (a), $W = \dfrac{GMm}{R}$. The initial kinetic energy supplies the needed work, so

$\frac{1}{2}mv_0^2 = \dfrac{GMm}{R} \quad\Rightarrow\quad v_0 = \sqrt{\dfrac{2GM}{R}}$

Handout H:

1) Slice water into horizontal disks. Work done on disk of height y, if $y=0$ is the bottom, is $\underbrace{\pi(5^2-y^2)}_{\text{Area}}\ \underbrace{\Delta y}_{\text{thickness}} \cdot \underbrace{1000}_{\text{density}}\ \underbrace{9.8}_{\text{gravity}}\ \underbrace{(7-y)}_{\text{distance}}$

$$\text{So } \pi\int_0^5 9800(175+y^3-7y^2-25y)\,dy$$

$$= 9800\pi\left(\tfrac{1}{4}y^4-\tfrac{7}{3}y^3-\tfrac{25}{2}y^2+175y\right)\Big]_0^5 = \boxed{4.2\cdot 10^5\,\pi\,J}$$