Mathematics 1b - Solution Set

Do: §8.3 #5,9

5) $\sum_{n=1}^{\infty} n^b$ is a p-series with $p = -b$. $\sum_{n=1}^{\infty} b^n$ is a geometric series. B y statment 1 in §8.3, the p-series is convergent if $p > 1$. In this case, $p = -b$, so $-b > 1 \Rightarrow b < -1$ a re the values for which the series converge. A geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $||r| < 1$, so $\sum_{n=1}^{\infty} b^n$ converges if $|b| < 1 \Rightarrow -1 < b < 1$.

9) $\frac{1}{n^2+n+1} < \frac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$ converg es by comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

Do: §8.4 #2, 12, 13, 19, (for 19, explain your reasoning clearly. There are seve ral different lines of reasoning that can be used.)

2)a) Since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 8 > 1$, part (b) of the Ratio Test tells us that th e series $\sum a_n$ is divergent.

b) Since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent, and therefore convergent.

c) Since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge OR diverge.

12) The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^4}$ satisfies (a) of the Alternating Series Test bec ause $\frac{1}{(n+1)^4} < \frac{1}{n^4}$ for all n, and (b) because $\lim_{n\to\infty} \frac{1}{n^4} = 0$, so the series converges. For the error part of the question, remember that (for an alternating series) th e limit of convergence is always between S_n and S_{n+1}, and that the difference between S_n and the limit of convergence is always $< b_{n+1}$ (where $b_n = |a_n|$). Therefore, since $b_5 = \frac{1}{5^4} = 0.0016 > 0.001$ and $b_6 = \frac{1}{6_4} \approx 0.00077 < 0.001$, so by the Alternating Series Estimation Thm, $n = 5$.

13) By the Ratio Test with the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{n!}$,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |a_{n+1} \cdot \frac{1}{a_n}| = \lim_{n\to\infty} |\frac{(-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2)^n}| = \lim_{n\to\infty} |\frac{-2}{n+1}| = 0 < 1. \quad (1)$$

So, the series is absolutely convergent (and convergent). As above, since $b_7 \approx 0.025 > 0.01$, and $b_8 \approx 0.006 < 0.01$, $n = 7$.

19) For an alternating series $\sum a_n$ to be absolutely convergent, it's counterpart positive series $\sum b_n = \sum |a_n|$ must be convergent. Since $\sum_{n=1}^{\infty} |\frac{(-1)^{n-1}}{n^{\frac{1}{2}}}| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$, and the latter is a divergent p-series ($p = \frac{1}{2} \leq 1$), the series in question is NOT absolutely convergent. NOTE: There may be other methods of doing this... basically, they need to show that either the given series or it's absolute value counterpart is divergent.

Plus: (i)alternating
(ii)the magnitude of the terms is decreasing
(iii)the magnitude of the terms tends to zero

Provide an Example of a diverging series that satisfies (i) and (ii), but not (iii).
Provide an Example of a diverging series that satisfies (ii) and (iii), but not (i).

There will be many varied answers for this problem. Just make sure they fit the conditions.
The harmonic series is a good example for (ii) and (iii). $(-1)^n * \frac{n+1}{n}$ satisfies (i) and (ii).

Plus: What exactly do we mean when we write $\sum a_k = 4$?

We mean that $lim_{n\leftarrow\infty} s_n = 4$, where s_n is the nth partial sum of the series. That is, the sequence of the partial sums of a_n has a limit, which is equal to 4.