0.1 Problem Set 5

Read §8.5.
Do: §8.4 #20, 22, 31, 33 plus Look back at #33. Which grows faster, exponentials or factorials? Tie your a nswer in with #33.

20) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has only positive terms, and by the ratio test,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(n=1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}| = \lim_{n\to\infty} (1+\frac{1}{n})^2 \cdot \frac{1}{2} = \frac{1}{2} < 1. \quad (1)$$

So, the series absolutely converges by the Ratio Test.

22) Again, using the Ratio Test,

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{(-3)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-3)^n}| = 3 \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1. \quad (2)$$

So, the series absolutely converges by the Ratio Test.

31)a) $\lim_{n\to\infty} |\frac{1}{(n+1)^3} \cdot \frac{n^3}{1}| = \lim_{n\to\infty} \frac{n^3}{(n+1)^3} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^3} = 1$. Inconclusive.
b) $\lim_{n\to\infty} |\frac{(n=1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}| = \lim_{n\to\infty} (1+\frac{1}{n})^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$. Conclusive (convergent).
c) $\lim_{n\to\infty} |\frac{(-3)^n}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-3)^{n-1}}| = 3 \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 3 > 1$. Conclusive (divergent).
d) $\lim_{n\to\infty} |\frac{\sqrt{n+1}}{1+(n+1)^2} \cdot \frac{1+n^2}{\sqrt{n}}| = \lim_{n\to\infty} [\sqrt{1+\frac{1}{n}} \cdot \frac{\frac{1}{n^2}+1}{\frac{s}{n^2}+(1+\frac{1}{n})^2}] = 1$. Inconclusive.

33)a) $\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}| = \lim_{n\to\infty} |\frac{x}{n+1}| = |x| \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$. So, by the Ratio Test, the given series converges for all x.
b) By the reverse of the Nth Term Test, since $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, it MUST equal 0.
Looking Back: Lets say we had a^n and $n!$. If $n < a$, the exponential grows faster, but for n on (a, ∞), the $n!$ grows faster (because you're now multiplying it by numbers larger than a, tho you keep multiplying the exponential by exactly a). If we think about prob. 33, this MUST be true, because if the numerator grew faster than the denominator, the series would diverge, not converge. (It doesn't matter that the exponential grows faster on $x < a$, because you can subtract the beginning finite number of terms from a series and not affect the convergence.)

Do: §8.5, #2, 4, 6, 14, 16 (dont worry about the endpoints)

2)a) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is (i) 0 if the series converges only when $x = a$, (ii) ∞ if the series converges for all x, or (iii) a positive number R, such that the series converges if $|x-a| < R$, and diverges for $|x-a| > R$. (R can usually be found through the Ratio Test.)
b) The interval of convergence is all x for which the series converges. Corresponding to the previous part, the interval of convergence is (i) a, (ii) $(-\infty, \infty)$, and (iii) $a-R < x < a+R$. In case (iii) we must test the endpoints to see if they converge or not.

4) $\lim_{n\to\infty} |\frac{x^{n+1}}{n+2} \cdot \frac{n+1}{x^n}| = \lim_{n\to\infty} \frac{|x|}{1+\frac{1}{n+1}} = |x|$. By the Ratio Test, the series then converges when $|x| < 1$, so $R = 1$. Check endpoints: At $x = -1$, the series diverges (it's basically a harmonic series, or they can prove it by the Nth Root Test). At $x = 1$, the series converges (basically, the alternating harmonic series, or using the Alternating Series Test). So $I = (-1, 1]$.
6) $\lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}| = |x| \lim_{n\to\infty} (\frac{n}{n+1})^2 = |x| < 1$ for convergence by Ratio Test, so $R = 1$. If $x = \pm 1$, $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series ($p = 2 > 1$). So, $I = [-1, 1]$.

14) $\lim_{n\to\infty} |\frac{x^{2n+1}}{(2n+1)!} \cdot \frac{(2n-1)!}{x^{2n-1}}| = \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n)} = 0 < 1$ for all x. By the Ratio Test, the series converges for all x, so $R = \infty$ and $I = (-\infty, \infty)$.

16) $\lim_{n\to\infty} |\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-2)^n(x+3)^n}| = \lim_{n\to\infty} \frac{2|x+3|}{\sqrt{1+\frac{1}{n}}} = 2|x+3| < 1 \Rightarrow |x+3| < \frac{1}{2}$ so $R = \frac{1}{2}$. Without bothering about thendpoints' convergence, $I = (-\frac{7}{2}, -\frac{5}{2})$.

Plus:
What EXACTLY do we mean when we write $\sum_{n=0}^{\infty} a_n = 7$? Your answer can be brief, but must be precise and accurate. You will get full credit only if you use words correctly.

Answers may vary, but should be something like, "The partial sums of the infinite series a_n approach 7."