Solution for Handout M

1. Solve the following differential equations for $y(t)$. Give the general solution.

   a) $y'' + 6y' = 7y$
   $y(t) = C_1 e^{-7t} + C_2 e^t$

   b) $y'' + 6y' + 9y = 0$
   $y(t) = C_1 e^{-3t} + C_2 t * e^{-3t}$

   c) $y'' + 5y' + 6y = 0$
   $y(t) = C_1 e^{-2t} + C_2 e^{-3t}$

2. For each of the differential equations in the problem above, suppose that the initial conditions are $y(0) = -2$ and $y'(0) = 0$.

   (i) Use the initial conditions to find $y(t)$.

   a) $y(t) = C_1 e^{-7t} + C_2 e^t$
   $C_1 + C_2 = -2$
   $-7C_1 + C_2 = 0$
   $C_1 = -\frac{1}{4} \quad C_2 = -\frac{7}{4}$
   $y(t) = -\frac{1}{4}e^{-7t} + -\frac{7}{4}e^t$

   b) $y(t) = C_1 e^{-3t} + C_2 t * e^{-3t}$
   $C_1 = -2$
   $-3C_1 + C_2 = 0$
   $C_2 = -6$
   $y(t) = -2e^{-3t} - 6t * e^{-3t} = -e^{-3t}(2 + 6t)$

   c) $y(t) = C_1 e^{-2t} + C_2 e^{-3t}$
   $C_1 + C_2 = -2$
   $-2C_1 + -3C_2 = 0$
   $C_1 = -6 \quad C_2 = 4$
   $y(t) = -6e^{-2t} + 4e^{-3t}$

   (ii) Find $\lim_{t\to\infty} y(t)$.

   a) $-\infty$
   b) 0
   c) 0

3. Interpret $x(t)$ as the position of a mass on a spring at time t where $x(t)$ satisfies

$$x'' + 4x' + 3x = 0.$$

   Suppose the mass is pulled out, stretching the spring one unit from its equilibrium position, and given an initial velocity of +2 units per second.

   (a) Find the position of the mass at time t.
   $x(t) = -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t}$

   (b) Determine whether or not the mass ever crosses the equilibrium position of $x = 0$.
   $0 = -\frac{3}{2}e^{-3t} + \frac{5}{2}e^{-t}$
   $5e^{-t} = 3e^{-3t}$
   $\frac{e^{-t}}{e^{-3t}} = \frac{3}{5}$
   $e^{2t} = \frac{3}{5}$
   $t = \ln(\frac{3}{5})/2$, which is negative, so NO

   (c) When (at what time) is the mass furthest from its equilibrium position? Approximately how far from the equilibrium position does it get?
   $x'(t) = \frac{9}{2}e^{-3t} - \frac{5}{2}e^{-t}$
   $0 = \frac{9}{2}e^{-3t} - \frac{5}{2}e^{-t}$
   $9e^{-3t} = 5e^{-t}$
   $\frac{9}{5} = e^{2t}$
   $t = \ln(\frac{9}{5})/2$
   $x(\ln(\frac{9}{5})/2) = 1.242259987...$

4. (a) Suppose that $x(t) = C_1e^{at} + C_2e^{bt}$. Show that $x(t) = 0$ at most once. Find the value of t for which $x(t) = 0$ if such a value exists.
$0 = C_1e^{at} + C_2e^{bt}$
$-\frac{C_2}{C_1} = e^{(a-b)t}$
$\ln(-\frac{C_2}{C_1}) = (a-b)t$
$t = \ln(-\frac{C_2}{C_1})/(a-b)$

(b) Suppose that $x(t) = C_1e^{at} + C_2te^{at}$. Show that $x(t) = 0$ at most once. Find the value of t for which $x(t) = 0$ if such a value exists.
$0 = C_1e^{at} + C_2te^{at}$
$C_1 = -C_2t$
$t = -\frac{C_1}{C_2}$

(c) Conclude from parts (a) and (b) that if the characteristic equation of $x'' + bx' + cx = 0$ has either one real root or two real roots then the differential equation cannot model a mass at the end of a spring in the scenario that the mass oscillates back and forth around the equilibrium point.
If the equation has either one or two real roots, then the equation for $x(t)$ would resemble the equation in part (b) or (c), respectively. Neither equations can describe oscillating motion because such motion would repeatedly pass through the equilibrium point ($x(t) = 0$), not just once (or not at all).

5. Let's try to make sense of the expression $e^{(a+bi)t}$, that is, e raised to a complex number $a + bi$ where $i = \sqrt{-1}$. To do this, first observe that $e^{(a+bi)t} = e^{at} \cdot e^{bit}$, where a and b are real numbers. The part we must make sense of is e^{bit}. Use the Maclaurin Series for e^x to expand e^{bti}. Gather all terms with i and all terms without i. (Factor out the i from the terms with i.) Now rewrite e^{bit} in terms of familiar functions.

$e^{bit} = 1 + bit + \frac{1}{2}(bit)^2 + \frac{1}{6}(bit)^3 + \frac{1}{24}(bit)^4 + ...$
$e^{bit} = 1 + bit - \frac{1}{2}(bt)^2 - \frac{i}{6}(bt)^3 + \frac{1}{24}(bt)^4 + ...$
$e^{bit} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}(bt)^{2n} + \sum_{n=0}^{\infty}(-1)^n\frac{i}{(2n+1)!}(bt)^{2n+1}$
$e^{bit} = \cos(bt) + i\sin(bt)$

Given your work above, what's $e^{\pi i}$?

$e^{\pi i} = \cos(\pi) + i\sin(\pi i) = -1 + i(0) = -1$