Solutions for Handout N

1. Solve the following differential equations for $y(x)$.

   (a) $y'' - 9y' = 0$
   $y(t) = C_1e^{9t} + C_2$

   (b) $y'' - 9y = 0$
   $y(t) = C_1e^{3t} + C_2e^{-3t}$

   (c) $y'' + 9y = 0$
   $y(t) = C_1e^{3ti} + C_2e^{-3ti} = C_3\cos(3t) + C_4\sin(3t)$

   (d) $y'' - 9 = 0$
   $y(t) = 4.5t^2 + C_1t + C_2$

   (e) $y'' - 2y' - y = 0$
   $y(t) = C_1e^{(1+\sqrt{2})t} + C_2e^{(1-\sqrt{2})t}$

   (f) $y'' - 2y' + 2y = 0$ $y(t) = C_1e^{(1+i)t} + C_2e^{(1-i)t} = C_3e^t\cos(t) + C_4e^t\sin(t)$

2. Suppose that $x'' + bx' + cx = 0$ is used to model the position of a block at the end of a vibrating spring.

   (a) What can you say about the signs of b and c? Explain.
   c must be positive because the roots of the characteristic equation need to be imaginary in order for the differential equation to describe oscillating motion (see quesition 4 from the Handout I), which requires a positive c value.
   b must be positive as well. There are a few ways of showing this. One is that, when the mass is at its equilibrium point, $x = 0$, so $x'' = -bx'$. Since the acceleration (at the equilibrium point, friction and drag are the only forces at play) is going in the opposite direction as the velocity, the value of b must be positive if the two sides of the equation $x'' = -bx'$ are to have the same sign.

   (b) As long as friction plays a role, we expect that regardless of the initial conditions $\lim_{t\to\infty} x(t) = 0$. Explain how your answer to part (a) guarantees this.
   If b is positive, then the exponent for e will be negative $(-\frac{b}{2}t)$ in the general solution for $x(t)$ $(C_1e^{at}\cos(bt) + C_2e^{at}\sin(bt))$. This means that both terms will approach zero as t approaches ∞; thus the value of $x(t)$ will approach zero as well.

   Hint: it is necessary to do three different cases.
   I am not exactly sure what was meant by this. If you have a different explanation and answer to (a) and (b) that requires three different cases to be explained, then your method is probably what the course head, Robin, had in mind. Otherwise, I believe that the above logic is sound and would suffice.

3. Write a differential equation of the form $x'' + bx' + cx = 0$ such that if $x(0) = 1$ and $x'(0) = 2$ then $x(t)$ has the property that

   (a) $\lim_{t\to\infty} x(t) = 0$
   $x'' + 6x' + 9x = 0 \quad x(t) = e^{-3t} + 5te^{-3t}$

   (b) $\lim_{t\to\infty} x(t) = \infty$
   $x'' - 7x' + 12 = 0 \quad x(t) = 2e^{3t} - e^{4t}$

   (c) $\lim_{t\to\infty} x(t)$ does not exist.
   $x'' - 2x' + 4x = 0 \quad x(t) = e^t\cos(t\sqrt{3}) + \frac{\sqrt{3}}{3}e^t\sin(t\sqrt{3})$

   Note: there are not unique answers to these problems! - The above are just examples!