① $\int e^x\sqrt{1-e^{2x}}\,dx = \int\sqrt{1-u^2}\,du = \frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\operatorname{Arcsin} u + c = \boxed{\frac{e^x}{2}\sqrt{1+e^{2x}} + \frac{1}{2}\operatorname{Arcsin}(e^x) + c}$

$u = e^x,\ du = e^x dx$

② $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$u = x \quad dv = \sec^2 x\,dx$
$du = dx \quad v = \tan x$

$= \boxed{x\tan x - \ln|\sec x| + c}$

③ $\int\frac{4x+4}{2x+1}dx = \int\frac{4x+2}{2x+1}dx + \int\frac{2}{2x+1}dx = \int 2\,dx + \int\frac{2}{2x+1}dx$

$= 2x + \ln|2x+1| + C$

$\int 7xe^{x^2}dx = \frac{7}{2}\int e^u = \frac{7}{2}e^u + c$

$u = x^2$
$du = 2x\,dx$

$= \boxed{\frac{7}{2}e^{x^2} + c}$

$\int\frac{dx}{x^2+6x+9} = \int\frac{dx}{(x+3)^2}$ $\quad u = x+3,\ du = dx$ $\quad = \int\frac{1}{u^2}du = -u^{-1} + c = \boxed{\frac{-1}{x+3} + c}$

⑥ $\int_{-10}^{10}\sin(x^3)\,dx = 0$
(sine is an odd function; therefore, integration from $-a$ to $+a$ will be zero since $x^3$ is also odd)

⑦ $\int_1^9\sqrt{4+3x}\,dx = \frac{1}{3}\int_7^{31}u^{1/2}\,du = \frac{2}{9}u^{3/2}\Big|_7^{31}$

$u = 4+3x \quad x=1\ u=7$
$du = 3dx \quad x=9\ u=31$

$= \boxed{\frac{2}{9}\left[(31)^{3/2} - (7)^{3/2}\right]}$

$\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

$u = \sin x \quad dv = e^x dx$
$du = \cos x\,dx \quad v = e^x$

Again $u = \cos x \quad dv = e^x dx$
$du = -\sin x\,dx \quad v = e^x$

$2\int e^x\sin x\,dx = e^x\sin x - e^x\cos x$

$\int e^x\sin x\,dx = \boxed{\frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x + C}$

⑨ $\int x\ln x\,dx$

$u = \ln x \quad dv = x\,dx$
$du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$

$= \frac{1}{2}x^2\ln x - \int\frac{1}{2}x\,dx$

$= \boxed{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}$

⑩ $\int\sin^2(7x)\,dx = \frac{1}{7}\int\sin^2 u\,du = \frac{1}{7}\int\frac{1}{2} - \frac{1}{2}\cos 2u\,du$

$u = 7x$
$du = 7dx$

$= \frac{1}{14}\int du - \frac{1}{14}\int\cos 2u\,du = \frac{1}{14}x - \frac{1}{28}\cos v\,dv$

$v = 2u$
$dv = 2du$

$= \boxed{\frac{1}{14}x - \frac{1}{28}\sin(14x) + c}$

(11) $\displaystyle\int \frac{4x^3+3x^2+2x+1}{x^4+x^3+x^2+x+1}\,dx$ $\qquad u = x^4+x^3+x^2+x+1$

$du = (4x^3+3x^2+2x+1)\,dx$

$= \displaystyle\int \frac{1}{u}\,du = \ln|u|+c = \boxed{\ln|x^4+x^3+x^2+x+1| + C}$

(12) $\displaystyle\int \frac{6x+3}{(x-1)(x-2)(x-3)}\,dx$ Partial Fractions $= \frac{3}{2}\int \frac{1}{x-1}\,dx - 33\int\frac{1}{x-2}\,dx + \frac{57}{2}\int\frac{1}{x-3}\,dx$

$= \boxed{\frac{3}{2}\ln|x-1| - 33\ln|x-2| + \frac{57}{2}\ln|x-3| + C}$

(13) $\displaystyle\int x\cos(x^2)\,dx = \frac{1}{2}\int \cos u\,du$

$u = x^2$

$du = 2x\,dx$ $\qquad = \frac{1}{2}\sin u + C$

$= \boxed{\frac{1}{2}\sin(x^2) + C}$

(14) $\displaystyle\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx = \boxed{x^2\sin x + 2x\cos x - \sin x + C}$

by parts $u = x^2$ $\qquad u = x \quad dv = \sin x\,dx$

$du = 2x\,dx$ $\qquad du = dx \quad v = -\cos x$

$dv = \cos x\,dx$

$v = \sin x$

(15) $\displaystyle\int \cos(x^2)\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}\,dx = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!} + C}$

$\cos x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$

$\cos(x^2) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}$

(16) $\displaystyle\int \cos^2 x\,dx = \int \frac{1}{2}\,dx + \frac{1}{2}\int\cos 2x\,dx = \frac{1}{2}x + \frac{1}{4}\int\cos u\,du = \boxed{\frac{1}{2}x + \frac{1}{4}\sin 2x + C}$

$u = 2x$

$du = 2dx$

(17) $\displaystyle\int x^3\cos(x^2)\,dx = \frac{1}{2}\int u\cos u\,du$ $\qquad = \frac{1}{2}\left[u\sin u - \int \sin u\,du\right]$

$u = x^2$ $\qquad u' = u \quad dv = \cos u\,du$ $\qquad = \frac{1}{2}u\sin u + \frac{1}{2}\cos u + C$

$du = 2x\,dx$ $\qquad du' = du \quad v = \sin u$

$= \boxed{\frac{1}{2}x^2\sin(x^2) + \frac{1}{2}\cos(x^2) + C}$

18) $\int \cos^3 x\,dx = \cos^2 x \cdot \sin x + 2\int \sin^2 x \cdot \cos x\,dx + C = \cos^2 x \cdot \sin x + 2\int u^2\,du + C = \boxed{\cos^2 x \cdot \sin x + 2\frac{\sin^3}{3} + C}$

$u = \cos^2 x \quad dv = \cos x\,dx \qquad u = \sin x \quad du = \cos x\,dx$

$du = -2\cos x \sin x\,dx \quad v = \sin x$

19) $\int_1^e \ln x\,dx = x\ln x \Big|_1^e - \int_1^e dx = x\ln x - x \Big|_1^e = e - e - 0 + 1 = \boxed{1}$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

20) $\int (\ln x)^2\,dx = x(\ln x)^2 - \ln x - \int(\ln x - 1)\,dx + C = \boxed{x(\ln x)^2 - \ln x - x\ln x + 2x + C}$

$u = \ln x \quad dv = \ln x\,dx$

$du = \frac{1}{x}dx \quad v = x\ln x - x$

21) $\int \frac{(\ln x)^2}{x}dx = \int u^2\,du = \frac{u^3}{3} + C = \boxed{\frac{(\ln x)^3}{3} + C}$

$u = \ln x$

$du = \frac{1}{x}dx$

22) $\int_e^{e^3} \frac{1}{x\ln x}dx = \int_e^{e^3} \frac{1}{u}du = \ln u \Big|_e^{e^3} = \ln(\ln x)\Big|_e^{e^3} = \ln 3 - 0 = \boxed{\ln 3}$

$u = \ln x$

$du = \frac{1}{x}dx$

23) $\int x\ln(x^2+1)\,dx = \frac{1}{2}\int \ln u\,du = \frac{1}{2}(u\ln u - u) + C = \boxed{\frac{1}{2}\left[(x^2+1)\ln(x^2+1) - (x^2+1)\right] + C}$

$u = x^2+1$

$du = 2x\,dx$

24) $\int \frac{x^2-1}{x^2+1}dx = (x^2-1)\tan^{-1}x - 2\int x\tan^{-1}x\,dx + C = (x^2-1)\tan^{-1}x - (x^2+1)\tan^{-1}x + x + C = \boxed{-2\tan^{-1}x + x + C}$

$u = x^2-1 \quad dv = \frac{1}{x^2+1}$

$du = 2x\,dx \quad v = \tan^{-1}x\,dx$

25) $\int \frac{x^2+1}{x^2-1}dx = \int \frac{(x^2-1)+2}{(x^2-1)}dx = \int\left(1 + \frac{2}{x^2-1}\right)dx = \int\left(1 - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \boxed{x - \ln(x+1) + \ln(x-1) + C}$

$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow A(x-1) + B(x+1) = 2$

$A = -B \qquad -A + B = 2$

$A = -1 \qquad B + B = 2$

$B = 1$

$\int \frac{x}{x^4+1}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du \quad \frac{1}{2}\tan u + C = \boxed{\frac{1}{2}\tan^{-1}x^2 + C}$

$u = x^2$

$du = 2x\,dx$

7) $\int \frac{x}{\sqrt{1+x^2}}\,dx \quad \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \quad u^{1/2}+C = \boxed{\sqrt{1+x^2}+C}$

$u = 1+x$

$du \quad 2x\,dx$

8) $\int \frac{x}{\sqrt{9x^2-4}}\,dx \quad \frac{1}{18}\int \frac{1}{\sqrt{u}}\,du \quad \frac{}{9}u^{1/2}+C = \boxed{\frac{1}{9}\sqrt{9x^2-4}+C}$

$u \quad 9x^2-4$

$du \quad 18x\,dx$

9) $\int \frac{e^x}{e^{2x}-4}\,dx \quad \int \frac{1}{u^2-4}\,du = \int \left[\frac{1}{4(u+2)} - \frac{1}{4(u-2)}\right]du = \boxed{\frac{1}{4}(u+2) - \frac{1}{4}\ln(u-2) + C}$

$\frac{1}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2} \qquad A(u+2)+B(u-2)=1$

$A=-B \quad 2A-2B=1$

$B=-\frac{1}{4} \quad 4A=1$

$A=\frac{1}{4}$

$-e$

$du = e^x dx$

30) $\int \frac{e^{2x}}{e^{2x}-4}\,dx \quad \frac{1}{2}\int \frac{1}{u}\,du = \boxed{\frac{1}{2}\ln(e^{2x}-4)+C}$

$u = e^{2x}-4$

$du = 2e^{2x}dx$

31) $\int_0^{\ln 3} \frac{e}{\sqrt{e^x+4}}\,dx = \int_{x=0}^{x=\ln 3} \frac{1}{\sqrt{u}}\,du \quad 2\sqrt{u}\Big|_{x=0}^{x=\ln 3} \quad 2\sqrt{e^x+4}\Big|_0^{\ln 3} = \boxed{2\sqrt{7}-2\sqrt{5}}$

$u = e^x+4$

$du = e^x dx$

32) $\int \frac{x+5}{x^2-2x-3}\,dx \quad \int \frac{x+5}{(x-3)(x+1)}\,dx \quad \int\left(\frac{2}{x-3} + \frac{1}{x+1}\right)dx = \boxed{2\ln(x-3) - \ln(x+1) + C}$

$\frac{x+5}{(x-3)(x+1)} \quad \frac{A}{x-3} + \frac{B}{x+1}$

$x+5 \quad A(x+1) + B(x-3)$

$A+B=1 \qquad A-3B=5$

$A=1-B \qquad 1-B\ \ 3B=5$

$A \quad -(-1) \qquad -4B=4$

$A=2 \qquad B=-1$

3) $\int \frac{x^2+3}{x^2+1}\,dx \quad \int \frac{x^2+1}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx = \boxed{x + 2\arctan(x) + C}$

4) $\int \sqrt{4-x^2}\,dx \quad \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta = \int 2\cos\theta \cdot 2\cos\theta\,d\theta = \int 4\cos^2\theta\,d\theta = 2\int(\cos 2\theta + 1)\,d\theta$

$x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta + 1 = 2\cos^2\theta$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$= \sin 2\theta + 2\theta + C$

$= \boxed{\sin 2\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + 2\sin^{-1}\left(\frac{x}{2}\right) + C}$

35. $\int \arctan(5x)\,dx$ : $u=\arctan(5x)$ $\quad dv=dx$

$du=\frac{5}{1+25x^2}$ $\quad v=x$

$$\int \arctan(5x)\,dx = x\arctan(5x) - \int \frac{5x\,dx}{1+25x^2} = x\arctan(5x) - \frac{1}{10}\ln\left(1+25x^2\right)+C$$

36. $\int x^2 e^{x^3}\,dx$ : $u=x^3$

$du=3x^2\,dx$

$$\frac{1}{3}\int e^{x^3}\left(3x^2\right)dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u+C = \frac{1}{3}e^{x^3}+C$$

37. $5\int \sin(3x)\cos(3x)\,dx = 5\left(\frac{\sin^2 3x}{6}\right) = \frac{5}{6}\sin^2(3x)+C$

38. $3\int \tan(2x)\,dx = 3\int \frac{\sin(2x)\,dx}{\cos(2x)}$ : $u=\cos 2x$

$du=-2\sin(2x)\,dx$

$$-\frac{3}{2}\int \frac{-2\sin 2x\,dx}{\cos 2x} = -\frac{3}{2}\int \frac{du}{u} = -\frac{3}{2}\ln|u| = -\frac{3}{2}\ln|\cos 2x|+C$$

39. $\int e^{\tan x}\sec^2 x\,dx$ : $u=\tan x$

$du=\sec^2 x\,dx$

$$\int e^u\,du = e^u+C = e^{\tan x}+C$$

40. $\int \cos(x)\sin(2\sin x)\,dx$ : $u=2\sin x$

$du=2\cos x\,dx$

$$\frac{1}{2}\int 2\cos(x)\,\sin(2\sin x)\,dx = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u+C = -\frac{1}{2}\cos(2\sin x)+C$$

41. $-\int \frac{\ln|1+x|}{(1+x)^2}\,dx \quad : \quad w=(1+x)$
$dw = dx$

$+\int \frac{\ln|w|}{-w^2}\,dw \quad : \quad u=\ln|w| \qquad dv=-w^{-2}\,dw$
$du = \frac{dw}{w} \qquad v = w^{-1}$

$$\int \frac{\ln|w|}{-w^2}\,dw = \frac{\ln|w|}{w} - \int \frac{dw}{w}\times\frac{1}{w} = \frac{\ln|w|}{w} + \int \frac{dw}{-w^2} = \frac{\ln|w|}{w} + \frac{1}{w} + C = \frac{\ln|1+x|}{(1+x)} + \frac{1}{(1+x)} + C$$

42. $$\int \frac{x^3\,dx}{x^2+2x+1} = \int x-2+\frac{3x+2}{(x+1)^2}\,dx = \frac{x^2}{2}-2x+\int\frac{(3x+2)}{(x+1)^2}\,dx = \frac{x^2}{2}-2x+\int \frac{3}{x+1}-\frac{1}{(x+1)^2}\,dx =$$

$$= \frac{x^2}{2}-2x+3\ln|x+1|+(x+1)^{-1}+C$$

43. $$\int \frac{2\,dx}{x^2+2x+1} = \int \frac{2dx}{(x+1)^2} = \int 2(x+1)^{-2}\,dx = -2(x+1)^{-1}+C$$

44. $$\int \frac{dx}{x(x+2)} = \int \frac{1}{2x}-\frac{1}{2(x+1)}\,dx = \frac{1}{2}\ln|x|-\frac{1}{2}\ln|x+1|+C = \frac{1}{2}\ln\left|\frac{\cancel{x}}{x+1}\right|+C$$

45. $$\int \frac{x^2}{(1-9x^2)^{3/2}}\,dx = \frac{x}{9}(1-9x^2)^{-\frac{1}{2}} - \frac{1}{27}\int \frac{3\cdot dx}{\sqrt{1-9x^2}}$$

$u = x \qquad dv = \frac{x}{(1-9x^2)^{3/2}}\,dx$
$du = dx \qquad v = \frac{1}{9}(1-9x^2)^{-\frac{1}{2}}$

$$= \boxed{\frac{x}{9}(1-9x^2)^{-\frac{1}{2}} - \frac{1}{27}\arcsin(3x) + C}$$

46. $\int 3x\cos x\,dx$ : $u=3x$ $\quad dv=\cos x\,dx$

$du=3dx$ $\quad v=\sin x$

$$\int 3x\cos(x)\,dx = 3x\sin x - 3\int \sin x\,dx = 3x\sin x + 3\cos x + C$$

47. $\int x(x^2+2x)^{\frac{1}{2}} + (x^2+2x)\,dx = \int (1+x)(x^2+2x)^{\frac{1}{2}}\,dx$ : $u=(x^2+2x)$

$du=(2x+2)\,dx$

$$\frac{1}{2}\int (2+2x)(x^2+2x)^{\frac{1}{2}}\,dx = \frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{3}{2}}+C = \frac{1}{3}(x^2+2x)^{\frac{3}{2}}+C$$

48. $\int \cos(2x)e^{3x}\,dx$ : integration by parts done twice:

$$\int \cos(2x)\,e^{3x}\,dx = \frac{e^{3x}(3\cos 2x + 2\sin 2x)}{13} + C$$

49. $\int_1^e \frac{\sin(\ln(x))}{x}\,dx$ : $u=\ln x$

$du=\frac{dx}{x}$

$$\int_1^e \frac{\sin(\ln(x))}{x}\,dx = \int_0^1 \sin(u)\,du = -\cos(u)\Big|_0^1 = 1-\cos(1)$$

50. $\int \text{sen}(\ln(x))\,dx$ $\quad u = \sin(\ln(x)) \quad dv = dx$

$du \ \frac{\cos(\ln(x))}{x}dx \quad v = x$

$\sin(\ln(x))\,dx = x\,\text{sen}(\ln(x)) \quad \cos(\ln(x))\,dx$

$u \ \cos(\ln(x)) \quad dv = dx$

$du \ -\frac{\sin(\ln(x))}{x} \quad v = x$

$\int \sin(\ln(x))\,dx = x(\text{sen}(\ln(x)) \ \cos(\ln(x))) - \ \text{sen}(\ln(x))\,dx$

$\Rightarrow \ \text{sen}(\ln(x))\,dx \ \frac{x}{2}(\sin(\ln(x)) \ \cos(\ln(x))) + c$

51 $\int_0^4 e^{\sqrt{x}}\,dx \quad u \ (x)^{\frac{1}{2}}$

$du \ \frac{1}{2}(x)^{-\frac{1}{2}}dx$

$\int_0 e^{\sqrt{x}}\,dx \ 2\int_0 u e^u\,du \quad w = \quad dv = e\,du$

$dw = du \quad v \ e$

$2\int_0^2 u e\,du = 2(u e^u \ \ e\,du)\Big|_0^2 = 2(u e^u \ e^u)\Big|_0^2 = 2(2e \ e^2 \ 0 + 1) = 2(e^2 + 1)$