

Mathematics 1b - Solution Set for PS 0-3

Problem Set # 0

Part I. Graphing Primer

1. How are the graphs of $y = f(x)$ and $y = f(x - 2)$ related? If the zeros (roots) of $f(x)$ are at $x = 3, 7,$ and $10,$ what are the zeros of $f(x - 2)$?

$y=f(x-2)$ is $y=f(x)$ shifted to the right by two units. The zeros of $f(x-2)$ would be $x = 5, 9, 12.$

2. Look at the graphs on the bottom of the next page. Write a possible formula for each function. (There may be more than one correct answer to some of these problems.) Check your answer with a graphing calculator or a computer.

a) $y = -\frac{1}{3}x^2 + 3$

They must show some work to earn full credit to prove they did not just guess and check with the calculator. One way is to use the x-intercepts to figure out the factors are $(x-3)$ and $(x+3),$ dividing through by 3 to obtain the correct value when $x = 0.$ Another way is to plug in the points provided $(0,3), (3,0), (-3,0)$ into the generic polynomial $y = ax^2 + bx + c$ and solve the system of three equations to arrive at values for a, b, and c. Other methods are valid as long as they are consistent with your knowledge of parabolas.

b) $y = 2 * 2^{-x}$ (which was the actual graph) or $y = 2 * (anyconstant)^{-x}$

Work shown here would be nice, but unnecessary in my opinion.

NOTE: $y = (constant)^{-x} + 1$ is not correct because graph tapers to $y = 0,$ not $y = 1,$ as x approaches infinity.

c) $y = \ln(x + 1)$

For the remainder, work is ideal but not necessary - use your discretion.

d) $y = -2\cos(x)$

e) $y = \cos(x * \pi) + 1$ (or equivalent)

f) $y = 1 - 10^{-x}$ (actual) or $y = 1 - (constant)^{-x}$

g) $y = \sin(x^2)$

h) $y = 10^{-0.1*x} * \sin(x)$

3. What are characteristics of polynomials that distinguish them from exponential, trigonometric, and logarithmic functions.

Answers can vary. Accept anything that is reasonable. Most straight forward difference is that elements of polynomials are composed of a coefficient times a variable raised to some constant value, whereas the other types of functions do not. Also, differentiation of the non-polynomial functions can be cyclic (except logarithmic functions) - small point. As stated, use your judgment to decide how good the responses are.

Part II. Primer for Taylor Series

The first third of this course deals primarily with series and polynomial approxiations of functions; the second third deals with integration and its applications. You already should know how to find the equation of the tangent line to a differentiable curve at a given point and should be able to interpret an integral as the area under a curve. In this set of questions you'll review this material and start to see how Taylor polynomials are constructed.

Suppose we want to approximate $\ln 1.1.$

1. The natural logarithm can be defined as an integral: $\ln x = \int_1^x \frac{1}{t} dt.$

- (a) Sketch the graph of $y = \frac{1}{t}$ and shade the area under the graph that corresponds to $\ln 1.1$.

See Graph 1 below.

Note: Area under the curve between $t = 1$ and $t = 1.1$ should be shaded.

- (b) Find an upper bound for $\ln 1.1$ by approximating the shaded area with the area of a circumscribed rectangle.

Circumscribed rectangle has dimensions $0.1 \times 1 = 0.1$

- (c) Find a lower bound for $\ln 1.1$ by approximating the shaded area with the area of an inscribed rectangle.

Inscribed rectangle has dimensions $0.1 \times \frac{10}{11} = \frac{1}{11} = .0909\dots$

- (d) Now approximate $\ln 1.1$ by approximating the shaded area with the area of a trapezoid. (Notice that this is simply the average of the answers to (b) and (c).)

$$\text{Area} = \frac{(\text{leg1}) + (\text{leg2})}{2} * \text{height}$$

$$\text{Area} = \frac{1 + \frac{10}{11}}{2} * 0.1 = 0.0954545\dots$$

- (e) Suppose you wanted a greater degree of accuracy. Describe what you could do to approximate the shaded area more accurately. (You need not do any computations for this part of the question.)

Once again, discretion is allowed. Common answer will likely be split up the interval more and find the sum of the smaller trapezoid areas.

2. Tangent line approximation:

- (a) Find the equation of the line tangent to $f(x) = \ln(1+x)$ at the point $x = 0$.

$$f'(x) = \frac{1}{1+x}$$

$$f'(0) = 1$$

$$f(0) = 0$$

$$y' = x + 0$$

- (b) On the same set of axes, sketch the graph of $f(x) = \ln(1+x)$ and the line you found in part (a).

See Graph 2 Below.

- (c) Use your tangent line approximation to $f(x) = \ln(1+x)$ to approximate $\ln 1.1$. (Let $x = 0.1$.) From your picture in part (b), do you expect your approximation to be too large, or to be too small?

$$f'(0.1) = x = 0.1. \text{ It will be too large.}$$

- (d) Compare your approximations of $\ln 1.1$ from problems 1 and 2 with the value given by your calculator or a computer.

$$\ln 1.1 = 0.0953$$

Students should compare with approximations and note if the real value was indeed higher than lower bounds and lower than upper bounds derived so far.

It is possible to refine the approaches taken in both problems 1 and 2 to improve the approximations. In the next problem, we'll improve upon the tangent line approximation.

3. We know that the tangent line to a function at a point is the best linear approximation to that function at that point. We can improve upon the tangent line approximation by finding the quadratic of best fit at $x = 0$.

- (a) Show that the quadratic $y = x - \frac{x^2}{2}$ intersects the graph of $f(x) = \ln(1+x)$ at $x = 0$, that it has the same first derivative as $f(x) = \ln(1+x)$ at $x = 0$, and that it has the same second derivative as $f(x) = \ln(1+x)$ at $x = 0$.

$$g(x) = y = x - \frac{x^2}{2}$$

$$g(0) = 0 \quad f(0) = \ln(1) = 0$$

$$\begin{aligned}
 g'(x) &= 1 - x & f'(x) &= 1/(1+x) \\
 g'(0) &= 1 & f'(0) &= 1 \\
 g''(x) &= -1 & f''(x) &= -1/(1+x)^2 \\
 g''(0) &= -1 & f''(0) &= -1
 \end{aligned}$$

- (b) On the same set of axes, graph $f(x) = \ln(1+x)$, the tangent line approximation to $f(x) = \ln(1+x)$ at $x = 0$ and the quadratic approximation $y = x - \frac{x^2}{2}$. Zoom in around $x = 0$ and record your observations.

See Graph 3 Below.

- (c) Compare the values of $f(x) = \ln(1+x)$ and $x - \frac{x^2}{2}$ at $x = .1$

$$\begin{aligned}
 f(.1) &= \ln 1.1 = .0953 \\
 g(.1) &= .1 - 0.1^2/2 = 0.095
 \end{aligned}$$

