Problem Set 10

§5.5 : # 4, 5, 8, 10, 14, 16, 20, 24, 27, 28, 32

4. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}\,dx$, so $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) =$
$2(-\cos u) + C = \boxed{-2\cos\sqrt{x} + C.}$

5. $\int \frac{4}{(1+2x)^3}\,dx$ Let $u = 1+2x$. Then $du = 2\,dx$, so $\int \frac{4}{(1+2x)^3}\,dx = 4\int u^{-3}\left(\frac{1}{2}du\right) =$
$2\,\frac{u^{-2}}{-2} + C = -\frac{1}{u^2} + C = \boxed{-\frac{1}{(1+2x)^2} + C.}$

8. $\int x e^{x^2}\,dx$ Let $u = x^2$. Then $du = 2x\,dx$, so $\int x e^{x^2}dx = \int e^u\left(\frac{1}{2}du\right) =$
$\frac{1}{2}e^u + C = \boxed{\frac{1}{2}e^{x^2} + C.}$

10. $\int x^3(1-x^4)^5\,dx$ Let $u = 1-x^4$. Then $du = -4x^3dx$, so $\int x^3(1-x^4)^5dx =$
$\int u^5\left(-\frac{1}{4}du\right) = -\frac{1}{4}\left(\frac{1}{6}u^6\right) + C = \boxed{-\frac{1}{24}(1-x^4)^6 + C.}$

14. $\int \frac{x}{x^2+1}\,dx$ Let $u = x^2+1$. Then $du = 2x\,dx$, so $\int \frac{x}{x^2+1}\,dx = \int \frac{\frac{1}{2}du}{u} =$
$\frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C = \boxed{\frac{1}{2}\ln(x^2+1) + C}$ since $[x^2+1>0]$
or $\boxed{\ln\sqrt{x^2+1} + C}$.

16. $\int t^2\cos(1-t^3)\,dt$ Let $u = 1-t^3$. Then $du = -3t^2dt$, so $\int t^2\cos(1-t^3)\,dt =$
$\int \cos u\left(-\frac{1}{3}du\right) = -\frac{1}{3}\sin u + C = \boxed{-\frac{1}{3}\sin(1-t^3) + C}.$

20. $\int \frac{\tan^{-1}x}{1+x^2}\,dx$ Let $u = \tan^{-1}x$. Then $du = \frac{dx}{1+x^2}$, so $\int \frac{\tan^{-1}x}{1+x^2}\,dx =$
$\int u\,du = \frac{u^2}{2} + C = \boxed{\frac{(\tan^{-1}x)^2}{2} + C}.$

24. $\int \frac{\cos(\pi/x)}{x^2}\,dx$ Let $u = \pi/x$. Then $du = \frac{-\pi}{x^2}\,dx$, so $\int \frac{\cos(\pi/x)}{x^2}\,dx =$
$\int \cos u\left(-\frac{1}{\pi}du\right) = -\frac{1}{\pi}\sin u + C = \boxed{-\frac{1}{\pi}\sin\frac{\pi}{x} + C}.$

27. $\displaystyle\int \frac{e^x+1}{e^x}\,dx = \int (1+e^{-x})\,dx = \boxed{x - e^{-x} + C}$ [Substitute $u=-x$].

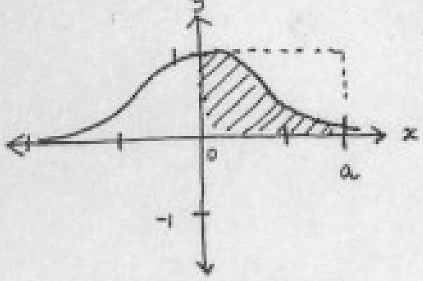
28. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$ Let $u = e^x+1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} =$

$\ln|u| + C = \boxed{\ln(e^x+1)+C}$.

32. $\displaystyle\int \frac{x}{1+x^4}\,dx$ Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} =$

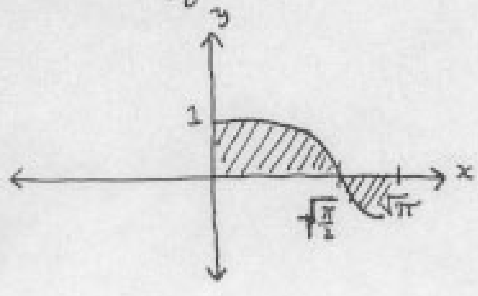
$\frac{1}{2}\tan^{-1} u + C = \boxed{\frac{1}{2}\tan^{-1}(x^2)+C}$.

• Claim 1 $\displaystyle 0 < \int_0^a e^{-x^2}\,dx < a$



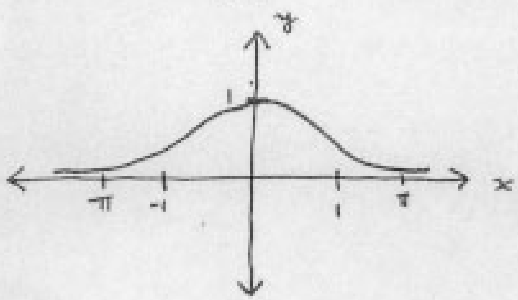
The rectangle has an area of a, since its base has a length of a and it has a height of 1. Clearly, the rectangle has a greater area than $\int_0^a e^{-x^2}\,dx$, so the claim is $\boxed{\text{TRUE}}$.

• Claim 2 $\displaystyle\int_0^{\sqrt{\pi}} \cos(x^2)\,dx < 0$



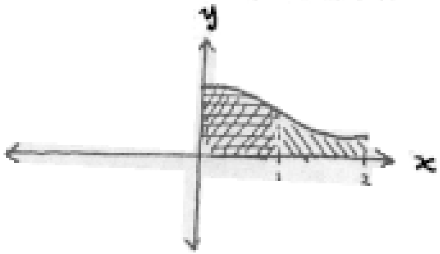
The part of the integral from $0 \to \sqrt{\pi/2}$ is positive, and larger in area than the integral from $\sqrt{\pi/2} \to \sqrt{\pi}$. So, the net result is clearly positive. Therefore, the claim is $\boxed{\text{FALSE}}$.

• Claim 3 $\displaystyle\int_{-\pi}^{\pi} e^{-x^2/\sqrt{2}}\,dx = 2\int_0^{\pi} e^{-x^2/\sqrt{2}}\,dx$



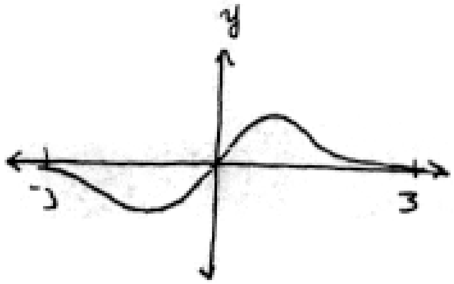
Because the integration is symmetric about the y-axis, and both $\int_{-\pi}^{0}$ and $\int_0^{\pi}$ are positive, we do indeed have that $\int_{-\pi}^{\pi} = 2\int_0^{\pi}$. The claim is $\boxed{\text{TRUE}}$.

• Claim 4: $\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx < \int_1^2 \frac{1}{\sqrt{1+x^4}}\,dx$



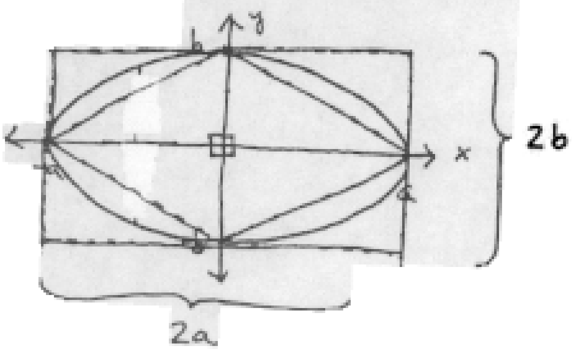
Clearly, the area under the curve from $0 \to 1$ is greater than the area under the curve from $1-2$ So, the claim is [redacted] $\boxed{\text{FALSE}}$.

Claim 5: $\int_{-3}^{3} \frac{x}{+x^4}\,dx > 0.001$



is symmetric, and one $\frac{1}{2}$ of the integral measures an area above the x-axis and the other $\frac{1}{2}$ of the integral measures an area below the x-axis $\left(\int_{-3}^{0}\right)$, we have that the two halves cancel out, and the integral is 0. Since 0.001, the claim is $\boxed{\text{FALSE}}$.

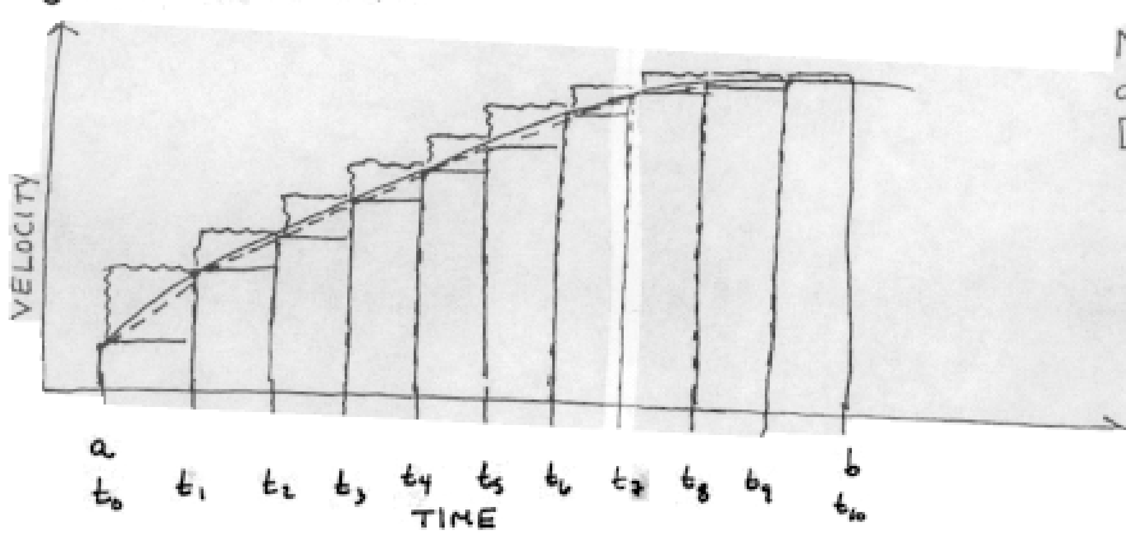
• Claim 6 The area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} =$ is greater than 2ab and less than 4ab



Clearly the rectangle of area 4ab is larger than the ellipse. Moreover, the area of the right triangles, collectively, is $4 \times \frac{1}{2}ab = 2ab$ Therefore, the area of the ellipse is A, with $2ab < A < 4ab$.

The claim is $\boxed{\text{true}}$.

(Diagram for Claims 7–8).



Note: the distance travelled on $[a, b] = \int_a^b v(t)$.

[redacted]

• Claim 7: $\sum_{k=1}^{10} v(t_{k-1})\,\Delta t$ > the distance travelled on [a, b]

$$\sum_{k=1}^{10} v(t_{k-1})\Delta t = \left[v(t_0) + v(t_1) + v(t_2) + \cdots v(t_9)\right]\Delta t$$

This represents the rectangles inscribed under the curve, which is clearly less than $\int_a^b v(t)$ So, the claim is $\boxed{\text{FALSE}}$.

• Claim 8: $\sum_{k=1}^{10} v(t_k)\,\Delta t$ > the distance travelled on [a, b]

$$\sum_{k=1}^{10} v(t_k)\Delta t = \left[v(t_1) + v(t_2) + v(t_3) + \cdots + v(_{10})\right]\Delta t$$

This represents the rectangles circumscribed above the curve, which is clearly geater than $\int_a^b v(t)$. So, the claim is $\boxed{\text{TRUE}}$.

laim 9: $\frac{}{2}\left[\sum_{k=1}^{10} v(t_k\ \Delta t + \sum_{k=1}^{10} v(t_{k-1})\Delta t\right]$

This is $\frac{}{2}\left[v(t_1) + v(t_2) + v(t_3) + \cdots + v(t_{10}) + v(t_0) + v(t_1) + \cdots + v(t_9)\right]\Delta t$

which is $\frac{}{2}\left[v(t_0) + 2v(t_1) + \cdots + 2v(t_9) + v(t_{10})\right]\Delta t$ This is the trapezoidal rule! Looking at the graph, we see that this sum is less than the integral from $a \to b$ (i.e, the distance travelled

So, the claim is $\boxed{\text{TRUE}}$.

§ 5.5 : # [redacted], 60

§ 5.6 : # 4, 5, 8, 14, 22, 28

§ Chapter 5 Review : # 5

§ 5.6 : # 34 (EXTRA-CREDIT)

60. Number of calculators $= x(4) - x(2) = \int_2^4 5000\left[1 - 100(t+10)^{-2}\right] dt =$

$5000\left[t + 100(t-10)^{-1}\right]_2^4 = 5000\left[\left(4 + \frac{100}{14}\right) - \left(2 + \frac{100}{12}\right)\right] \approx \boxed{4048}$

4. $\int x^4 \ln x \, dx$ Let $u = \ln x$, $dv = x^4\,dx \Rightarrow du = (1/x)\,dx$, $v = \frac{1}{5}x^5$. Then

$\int x^4 \ln x \, dx = 1/5\, x^5 \ln x - \int 1/5\, x^5 (1/x)\, dx =$ [redacted] $1/5\, x^5 \ln x - 1/5 \int x^4\, dx =$

$1/5\, x^5 \ln x - 1/5\,(1/5\, x^5) + C = \boxed{1/5\, x^5 \ln x - \frac{1}{25}x^5 + C}$ or $\boxed{\frac{1}{25}x^5(5\ln x - 1) + C}$.

5. $\int x \sin 4x \, dx$ Let $u = x$, $dv = \sin 4x\, dx \Rightarrow du = dx$, $v = -\frac{1}{4}\cos 4x$ Then

by Equation 2, $\int x \sin 4x\, dx = -\frac{1}{4}x\cos 4x - \int\left(-\frac{1}{4}\cos 4x\right)dx = -\frac{1}{4}x\cos 4x +$

$\frac{1}{4}\left(\frac{1}{4}\sin 4x\right) + C = \boxed{-\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C.}$

8. $\int x^2 \sin ax\, dx$ First let $u = x^2$, $dv = \sin ax\, dx \Rightarrow du = 2x\,dx$, $v = -\frac{1}{a}\cos ax$.

Then, by Equation 2, $I = \int x^2 \sin ax\, dx = -\frac{x^2}{a}\cos ax - \int\left(-\frac{1}{a}\right)\cos ax\,(2x\,dx) =$

$-\frac{x^2}{a}\cos(ax) + \frac{2}{a}\int x\cos ax\, dx$. Next let $U = x$, $dV = \cos ax\, dx \Rightarrow$

$dU = dx$, $V = \frac{1}{a}\sin ax$. So $\int x \cos ax\, dx = \frac{x}{a}\sin ax - \int \frac{1}{a}\sin ax\, dx =$

$\frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C_1$. Substituting for $\int x\cos ax\, dx$, we get

$I = -\frac{x^2}{a}\cos ax + \frac{2}{a}\left(\frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C_1\right) =$

$\boxed{-\frac{x^2}{a}\cos ax + \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax + C}$.

14. $\int e^{-\theta}\cos 2\theta\, d\theta$ First let $u = e^{-\theta}$, $dv = \cos 2\theta\, d\theta \Rightarrow du = -e^{-\theta}d\theta$, $v =$
$\frac{1}{2}\sin 2\theta$. Then $I = \int e^{-\theta}\cos 2\theta\, d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\,(-e^{-\theta}d\theta) =$

$\frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\, d\theta$. Next let $U = e^{-\theta}$, $dV = \sin 2\theta\, d\theta \Rightarrow$
$dU = -e^{-\theta}d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so $\int e^{-\theta}\sin 2\theta\, d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta(-e^{-\theta}d\theta)$

$= -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\, d\theta$. So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] =$

$\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \Rightarrow \frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \Rightarrow$

$I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \boxed{\frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C}$.

22. $\int x\tan^{-1}x\, dx$ Let $u = \tan^{-1}x$, $dv = x\,dx \Rightarrow du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\int x\tan^{-1}x\, dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$. $\int\frac{x^2}{1+x^2}dx = \int\frac{(1+x^2)-1}{(1+x^2)}dx =$

$\int 1\,dx - \int\frac{1}{1+x^2}dx = x - \tan^{-1}x + C_1$. So $\int x\tan^{-1}x\, dx = \frac{1}{2}x^2\tan^{-1}x -$

$\frac{1}{2}(x - \tan^{-1}x + C_1) = \boxed{\frac{1}{2}(x^2\tan^{-1}x + \tan^{-1}x - x) + C}$.

28. $\int_1^4 e^{\sqrt{x}}\,dx$ Let $w=\sqrt{x}$, so that $x=w^2$ and $dx = 2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx =$ $\int_1^2 e^{w}\,2w\,dw$. Now use parts with $u = 2w$, $dv = e^{w}dw$, $du = 2dw$, $v = e^{w}$ to get $\int_1^2 e^{w}\,2w\,dw = \left[2we^{w}\right]_1^2 - 2\int_1^2 e^{w}\,dw = 4e^2 - 2e - 2(e^2-e) =$

$\boxed{2e^2}$

5. $\boxed{\text{FALSE}}$ For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

34. (a) Let $u = \cos^{n-1}x$, $dv = \cos x\,dx \Rightarrow du = -(n-1)\cos^{n-2}x\sin x\,dx$, $v = \sin x$ in (2):

$$\int \cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x\sin^2 x\,dx$$
$$= \cos^{n-1}x\sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$
$$= \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\,dx - (n-1)\int\cos^n x\,dx$$

Rearranging terms gives $n\int\cos^n x\,dx = \cos^{n-1}x\sin x + (n-1)\int\cos^{n-2}x\,dx$ or

$$\int\cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int\cos^{n-2}x\,dx$$

(b) Take $n=2$ in part (a) to get $\int\cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx =$

$\boxed{\frac{x}{2} + \frac{\sin 2x}{4} + C}$.

(c) $\int\cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int\cos^2 x\,dx = \boxed{\frac{1}{4}\cos^3 x\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C}$