Problem Set 18

2. (a) Since $y = 1/(2x-1)$ is defined and continuous on $[1,2]$, the integral is proper.

(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

6. $\displaystyle\int_2^{\infty} \frac{dx}{(x+3)^{3/2}} = \lim_{t\to\infty}\int_2^t \frac{dx}{(x+3)^{3/2}} = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{x+3}}\right]_2^t = \lim_{t\to\infty}\left[\frac{-2}{\sqrt{t+3}} + \frac{2}{\sqrt{5}}\right] = \frac{2}{\sqrt{5}}$. Convergent

12. $I = \int_{-\infty}^{\infty}(2-v^4)\,dv = I_1 + I_2 = \int_{-\infty}^{0}(2-v^4)\,dv + \int_0^{\infty}(2-v^4)\,dv$, but
$I_1 = \lim\limits_{t\to-\infty}\left[2v - \frac{1}{5}v^5\right]_t^0 = \lim\limits_{t\to-\infty}\left(-2t + \frac{1}{5}t^5\right) = -\infty$. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

13. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx$.
$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim\limits_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim\limits_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1-e^{-t^2}\right) = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$, and
$\int_0^{\infty} xe^{-x^2}\,dx = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2}$.
Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$. Convergent

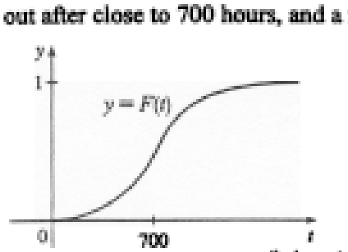
20. Since $f(r) = \dfrac{1}{r^2+4}$ is even,

$$I = \int_{-\infty}^{\infty} f(r)\,dr = 2\int_0^{\infty} f(r)\,dr = 2\lim_{t\to\infty}\int_0^t \frac{1}{r^2+4}\,dr = 2\lim_{t\to\infty}\left[\frac{1}{2}\arctan\frac{r}{2}\right]_0^t$$
$$= \lim_{t\to\infty}\left(\arctan\frac{t}{2} - 0\right) = \frac{\pi}{2}.\quad \text{Convergent}$$

24. There is an infinite discontinuity at the left endpoint of $[0,3]$.
$\displaystyle\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 \frac{dx}{x^{3/2}} = \lim_{t\to 0^+}\left[\frac{-2}{\sqrt{x}}\right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t\to 0^+}\frac{2}{\sqrt{t}} = \infty$. Divergent

26. $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\left[\frac{3}{2}(x-9)^{2/3}\right]_1^t = \lim_{t\to 9^-}\left[\frac{3}{2}(t-9)^{2/3} - \frac{3}{2}(4)\right] = 0 - 6 = -6.$
Convergent

53. (a) We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^{\infty} r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

Extra Credit:

50. (a) $n=0$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1 = 1$

$n=1$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$. To evaluate $\displaystyle\int xe^{-x}\,dx$, we'll use integration by parts with $u=x$, $dv=e^{-x}\,dx \;\Rightarrow\; du=dx$, $v=-e^{-x}$.

So $\displaystyle\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x} + C$ and

$\displaystyle\lim_{t\to\infty}\int_0^t xe^{-x}\,dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t$

$\displaystyle = \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$

$= 0-0+1$ [use l'Hospital's Rule] $=1$

$n=2$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$. To evaluate $\displaystyle\int x^2e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$\displaystyle\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx = \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx$

$= 0+0+2(1)$ [use l'Hospital's Rule and the result for $n=1$] $=2$

$n=3$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$

$= 0+0+3(2)$ [use l'Hospital's Rule and the result for $n=2$] $=6$

(b) For $n=1$, 2, and 3, we have $\displaystyle\int_0^\infty x^n e^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\displaystyle\int_0^\infty x^n e^{-x}\,dx = n!$.

(c) Suppose that $\displaystyle\int_0^\infty x^k e^{-x}\,dx = k!$ for some positive integer k. Then $\displaystyle\int_0^\infty x^{k+1}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$.

To evaluate $\displaystyle\int x^{k+1}e^{-x}\,dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}\,dx \;\Rightarrow\; du=(k+1)x^k\,dx$, $v=-e^{-x}$.

So $\displaystyle\int x^{k+1}e^{-x}\,dx = -x^{k+1}e^{-x}$