## 6.3

1. $y = 2 - 3x \;\Rightarrow\; L = \int_{-2}^{1} \sqrt{1 + (dy/dx)^2}\,dx = \int_{-2}^{1} \sqrt{1 + (-3)^2}\,dx = \sqrt{10}\,[1 - (-2)] = 3\sqrt{10}$.

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$$L = [\text{distance from } (-2, 8) \text{ to } (1, -1)] = \sqrt{[1 - (-2)]^2 + [(-1) - 8]^2} = \sqrt{90} = 3\sqrt{10}$$

4. $y = 2^x \;\Rightarrow\; dy/dx = (2^x)\ln 2 \;\Rightarrow\; L = \int_0^3 \sqrt{1 + (\ln 2)^2\, 2^{2x}}\,dx$

21. The prey hits the ground when $y = 0 \;\Leftrightarrow\; 180 - \frac{1}{45}x^2 = 0 \;\Leftrightarrow\; x^2 = 45 \cdot 180 \;\Rightarrow\; x = \sqrt{8100} = 90$, since $x$ must be positive. $y' = -\frac{2}{45}x \;\Rightarrow\; 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$\begin{aligned} L &= \int_0^{90} \sqrt{1 + \tfrac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1 + u^2}\left(\tfrac{45}{2}\,du\right) \quad \left[u = \tfrac{2}{45}x,\ du = \tfrac{2}{45}\,dx\right] \\ &\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\!\left(u + \sqrt{1 + u^2}\right)\right]_0^4 \\ &= \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln(4 + \sqrt{17})\right] = 45\sqrt{17} + \tfrac{45}{4}\ln(4 + \sqrt{17}) \approx 209.1 \text{ m} \end{aligned}$$

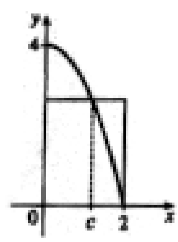
## 6.4

2. $g_{ave} = \frac{1}{4-1}\int_1^4 \sqrt{x}\,dx = \frac{1}{3}\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{9}\left[x^{3/2}\right]_1^4 = \frac{2}{9}(8 - 1) = \frac{14}{9}$

5. (a) 
$$\begin{aligned} f_{ave} &= \tfrac{1}{2-0}\int_0^2 (4 - x^2)\,dx \\ &= \tfrac{1}{2}\left[4x - \tfrac{1}{3}x^3\right]_0^2 \\ &= \tfrac{1}{2}\left(8 - \tfrac{8}{3}\right) = \tfrac{8}{3} \end{aligned}$$

(b) $f_{ave} = f(c) \;\Leftrightarrow\; \frac{8}{3} = 4 - c^2 \;\Leftrightarrow\; c^2 = \frac{4}{3}$
$\Leftrightarrow\; c = \frac{2}{\sqrt{3}} \approx 1.15$

(c)




11. Let $t = 0$ and $t = 12$ correspond to 9 A.M. and 9 P.M., respectively.

$$\begin{aligned} T_{ave} &= \tfrac{1}{12-0}\int_0^{12}\left[50 + 14\sin\tfrac{1}{12}\pi t\right]dt = \tfrac{1}{12}\left[50t - 14 \cdot \tfrac{12}{\pi}\cos\tfrac{1}{12}\pi t\right]_0^{12} \\ &= \tfrac{1}{12}\left[50 \cdot 12 + 14 \cdot \tfrac{12}{\pi} + 14 \cdot \tfrac{12}{\pi}\right] = \left(50 + \tfrac{28}{\pi}\right)\,^\circ\text{F} \approx 59\,^\circ\text{F} \end{aligned}$$

16. $$v_{ave} = \frac{1}{R - 0}\int_0^R v(r)\,dr = \frac{1}{R}\int_0^R \frac{P}{4\eta l}(R^2 - r^2)\,dr = \frac{P}{4\eta l R}\left[R^2 r - \tfrac{1}{3}r^3\right]_0^R = \frac{P}{4\eta l R}\left(\tfrac{2}{3}\right)R^3 = \frac{PR^2}{6\eta l}.$$

Since $v(r)$ is decreasing on $(0, R]$, $v_{max} = v(0) = \dfrac{PR^2}{4\eta l}$. Thus, $v_{ave} = \frac{2}{3}v_{max}$.

## 6.5

$$= 10\left[-\frac{1}{u}\right]_1^{10} = 10\left(-\tfrac{1}{10} + 1\right) = 9 \text{ ft-lb}$$

4. $25 = f(x) = kx = k(0.1)$ [10 cm = 0.1 m], so $k = 250$ N/m and $f(x) = 250x$. Now 5 cm = 0.05 m, so $W = \int_0^{0.05} 250x\,dx = \left[125x^2\right]_0^{0.05} = 125(0.0025) = 0.3125 \approx 0.31$ J.

7. The portion of the rope from $x$ ft to $(x + \Delta x)$ ft below the top of the building weighs $\frac{1}{2}\,\Delta x$ lb and must be lifted $x_i^*$ ft, so its contribution to the total work is $\frac{1}{2}x_i^*\,\Delta x$ ft-lb. The total work is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}x_i^*\,\Delta x = \int_0^{50} \tfrac{1}{2}x\,dx = \left[\tfrac{1}{4}x^2\right]_0^{50} = \tfrac{2500}{4} = 625 \text{ ft-lb}$$

Notice that the exact height of the building does not matter (as long as it is more than 50 ft).

10. The work needed to lift the bucket itself is 4 lb · 80 ft = 320 ft-lb. At time $t$ (in seconds) the bucket is $x_i^* = 2t$ ft above its original 80 ft depth, but it now holds only $(40 - 0.2t)$ lb of water. In terms of distance, the bucket holds $\left[40 - 0.2\left(\frac{1}{2}x_i^*\right)\right]$ lb of water when it is $x_i^*$ ft above its original 80 ft depth. Moving this amount of water a distance $\Delta x$ requires $\left(40 - \frac{1}{10}x_i^*\right)\Delta x$ ft-lb of work. Thus, the work needed to lift the water is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \left(40 - \tfrac{1}{10}x_i^*\right)\Delta x = \int_0^{80} \left(40 - \tfrac{1}{10}x\right)dx = \left[40x - \tfrac{1}{20}x^2\right]_0^{80} = (3200 - 320) \text{ ft-lb}$$

Adding the work of lifting the bucket gives a total of 3200 ft-lb of work.

12. A horizontal cylindrical slice of water $\Delta x$ ft thick has a volume of $\pi r^2 h = \pi \cdot 12^2 \cdot \Delta x$ ft$^3$ and weighs about $(62.5 \text{ lb/ft}^3)(144\pi\,\Delta x \text{ ft}^3) = 9000\pi\,\Delta x$ lb. If the slice lies $x_i^*$ ft below the edge of the pool (where $1 \le x_i^* \le 5$), then the work needed to pump it out is about $9000\pi x_i^*\,\Delta x$. Thus,

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 9000\pi x_i^*\,\Delta x = \int_1^5 9000\pi x\,dx = \left[4500\pi x^2\right]_1^5 = 4500\pi(25 - 1) = 108{,}000\pi \text{ ft-lb}$$

17. (a) $W = \displaystyle\int_a^b F(r)\,dr = \int_a^b G\frac{m_1 m_2}{r^2}\,dr = Gm_1m_2\left[\frac{-1}{r}\right]_a^b = Gm_1m_2\left(\frac{1}{a} - \frac{1}{b}\right)$

(b) By part (a), $W = GMm\left(\dfrac{1}{R} - \dfrac{1}{R + 1{,}000{,}000}\right)$ where $M$ = mass of earth in kg, $R$ = radius of earth in m, and $m$ = mass of satellite in kg. (Note that 1000 km = 1,000,000 m.) Thus,
$W = (6.67 \times 10^{-11})(5.98 \times 10^{24})(1000) \times \left(\dfrac{1}{6.37 \times 10^6} - \dfrac{1}{7.37 \times 10^6}\right) \approx 8.50 \times 10^9$ J.

18. (a) $W = \displaystyle\int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm\lim_{t\to\infty}\left(\frac{-1}{t} + \frac{1}{R}\right)$

$= \dfrac{GMm}{R}$, where $M$ = mass of earth $= 5.98 \times 10^{24}$ kg, $m$ = mass of satellite $= 10^3$ kg, $R$ = radius of earth $= 6.37 \times 10^6$ m, and $G$ = gravitational constant $= 6.67 \times 10^{-11}$ N-m$^2$/kg$^2$. Therefore,
work $= \dfrac{6.67 \times 10^{-11} \cdot 5.98 \times 10^{24} \cdot 10^3}{6.37 \times 10^6} \approx 6.26 \times 10^{10}$ J.

(b) From part (a), $W = \dfrac{GMm}{R}$. The initial kinetic energy supplies the needed work, so

$\frac{1}{2}mv_0^2 = \dfrac{GMm}{R} \Rightarrow v_0 = \sqrt{\dfrac{2GM}{R}}$

Handout H:

1) Slice water into horizontal disks. Work done on disk of height y, if y=0 is the bottom, is $\underbrace{\pi(5^2-y^2)}_{\text{Area}}\ \underbrace{\Delta y}_{\text{thickness}} \cdot \underbrace{1000}_{\text{density}}\ \underbrace{9.8}_{\text{gravity}}\ \underbrace{(7-y)}_{\text{distance}}$

$$\text{So } \pi\int_0^5 9800(175+y^3-7y^2-25y)\,dy$$

$$= 9800\pi\left(\tfrac{1}{4}y^4-\tfrac{7}{3}y^3-\tfrac{25}{2}y^2+175y\right)\Big]_0^5 = \boxed{4.2\cdot 10^5\,\pi\,J}$$