# *Mathematics 1b - Solution Set 3*

Do: §8.3 # 1, 2, 3, 4, 10, 19, 20

1) Graph of the picture shows $a_n$ as the area of shaded boxes each under the curve, with width of 1 and height of $a_n$ (thus the area is the value $a_n$). Therefore, $\sum_{n=2}^{\infty} \frac{1}{n^{1.3}}$ (i.e., the sum of the areas of all these boxes) $< \int_1^{\infty} \frac{1}{x^{1.3}} dx$. The integral converges with $p = 1.3 > 1$, so the series converges.

2) The figures for the two series are as follows: $\sum_{i=1}^{5} a_i$ is the area of the shaded boxes with width 1 and height $a_1, a_2, ..., a_5$. This spans the interval on the graph from $x = 1$ to $x = 6$. $\sum_{i=2}^{6} a_i$ is the set of shaded boxes with width 1 and height $a_2, a_3, ..., a_6$ - they all lie under the graph of $f(x)$. By looking at these graphs, we can see that the sum of areas of the first set of shaded boxes (and therefore the sum of the first 5 terms of $a_i$) is greater than the area under the curve (the integral of $f(x)$), while the areas of the second set of shaded boxes summates to a value less than the integral. Therefore, $\sum_{i=1}^{5} a_i > \int_1^6 f(x)dx > \sum_{i=2} 6a_i$.

3) a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all $n$ and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent.
b) If $a_n < b_n$ for all $n$, then $\sum a_n$ is convergent.

4) a) If $a_n > b_n$ for all $n$, then $\sum a_n$ is divergent.
b) We cannot say anything about $\sum a_n$ for similar reasons as in question 3a.

10) $\frac{1}{2n-1} > \frac{1}{2n} = \frac{1}{2} * \frac{1}{n}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges because it is a nonzero contant multiple of the divergent harmonic series.

19) $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$ for all $ngeq1$, so $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

20) $\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = (\frac{3}{2})^n$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} (\frac{3}{2})^n$.

Plus:

*Problem on p−series*

In this problem you will learn about a family of series known as $p-$series. This is an important problem, and we will use the result of your work repeatedly both in class and on exams.

A $p-$series is a series of the form

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots.$$

1. If $p < 0$ then the series diverges by the $n$th term test. Explain.

   The series would be the same as $\sum_{n=1}^{\infty} n^{-p}$ (with $-p$ being a positive value), which diverges because of the $n$th term, or divergence, test ($\lim_{n \longrightarrow \infty} n^{positivevalue} \neq 0$).

2. Show that if $p > 1$ then $\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx$ is finite.

   $\lim_{b \longrightarrow \infty} \int_1^b \frac{1}{x^p} = \lim_{b \longrightarrow \infty} \frac{-1}{(p-1)x^{p-1}}|_1^b = \lim_{b \longrightarrow \infty} \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)(1)}$. If $p > 1$, the definite integral is finite because $b$ remains in the denominator and causes the first term to approach zero as $b$ approaches infinity. Thus, if $p > 1$, the definite integral is finite.

   Show that if $p = 1$ then $\lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx = \infty$.

   If $p = 1$, then the integral would be $\lim_{b \longrightarrow \infty} lnx|_1^b$, which is infinite because $\lim_{b \longrightarrow \infty} lnb = \infty$.

   Show that if $0 < p < 1$ then $\lim_{b \to \infty} \int_1^b \frac{1}{x^p} dx = \infty$.

   If $0 < p < 1$, then $p - 1 < 0$ and $\lim_{b \longrightarrow \infty} \frac{1}{b^{p-1}} = \infty$. Therefore, $\lim_{b \longrightarrow \infty} \int_1^b \frac{1}{x^p} = \lim_{b \longrightarrow \infty} \frac{-1}{(p-1)x^{p-1}}|_1^b = \lim_{b \longrightarrow \infty} \frac{-1}{(p-1)b^{p-1}} - \frac{-1}{(p-1)(1)} = \infty -$(a finite number)$= \infty$.

   Conclude that $\int_1^{\infty} \frac{1}{x^p} dx$ diverges for $0 < p < 1$ and converges for $p > 1$.

   Based on the Integral Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ because $\int_1^{\infty} \frac{1}{x^p}$ converges for those values of $p$. Similarly, the series diverges when $0 < p < 1$ becuase the integral diverges for those values of $p$.

3. Conclude from your work in parts (a) and (b) that $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ converges if $p > 1$ and diverges if $p < 1$.

   We showed above that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $0 < p < 1$. Using the same argument, the series also diverges if $p < 0$, and if $p = 0$ the series becomes $\sum 1$, which diverges. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p < 1$ and converges if $p > 1$. We showed that diverges if $p < 1$.