Problem Set 5

Read §8.7 Do: §8.7, #9, 11, 18, 20, 23, 27, 29, 36

9) $f^{(n)}(x) = e^x$, so $f^{(3)} = e^3$ and $e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n$. If $a_n = \frac{e^3}{n!}(x-3)^n$, then $\lim_{n\longrightarrow\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\longrightarrow\infty} |\frac{e^3(x-3)^{n+1}}{(n+1)!}\frac{n!}{e^3(x-3)^n}| = \lim_{n\longrightarrow\infty} \frac{|x-3|}{n+1} = 0 < 1$ for all x, so $R = \infty$.

11) $f^{(n)}(1) = (-1)^n n!$, and $\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$. Therefore, $a_n = (-1)^n(x-1)^n$ and $\lim_{n\longrightarrow\infty} |\frac{a_{n+1}}{a_n}| = |x-1| < 1$ for it to converge. Thus, $R = 1$.

18) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow f(x) = e^{-x/2} = \sum_{n=0}^{\infty} \frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}x^n$; $R = \infty$.

20) $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \longrightarrow f(x) = \sin(x^4) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{8n+4}$, $R = \infty$.

23) $\sin^2 x = \frac{1}{2}[1 - \cos(2x)] = \frac{1}{2}[1 - \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n}}{(2n)!} = 2^{-1}[1 - 1 - \sum_{n=0}^{\infty} \frac{(-1)^n(2x)^{2n}}{(2n)!}] = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$, so $R = \infty$.

27) $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} \longrightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n(x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$. With the graph, notice that as n increases, T_n approximates $f(x)$ more closely.

29) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!}(0.2)^2 - \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 - \frac{1}{5!}(0.2)^5 + \frac{1}{6!}(0.2)^6 - ....$ Because $\frac{1}{6!}(0.2)^6 = 8.88...x10^{-8}$, $e^{-0.2} \approx \sum_{n=0}^{5} \frac{(-0.2)^n}{n!} \approx 0.81873$, which is correct to 5 decimal places based on the Alternating Series Estimation Theorem.

36) $\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n(x^2)^{2n}}{(2n)!}$, so $\int_0^0 .5\cos(x^2)dx = \int_0^0 .5\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}dx = \sum_{n=0}^{\infty}[\frac{(-1)^n x^{4n+1}}{(4n+1)(2n)!}]_0^0.5 = 0.5 - \frac{(0.5)^5}{5\cdot 2!} + \frac{(0.5)^9}{9\cdot 4!} - ....$ $\frac{(0.5)^9}{9\cdot 4!} \approx 0.000009$, so $\int_0^0 .5\cos(x^2)dx \approx 0.5 - \frac{(0.5)^5}{5\cdot 2!} \approx 0.497$ with accuracy up to three decimal places based on the Alternating Series Estimation Theorem.