PROBLEM SET #9 (SECTION 8.8)

8.7: 25.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$1/2$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$

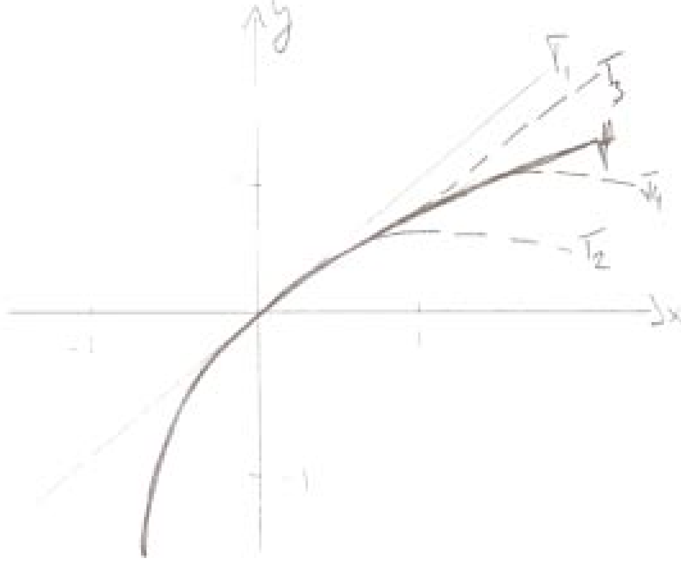
$$\Rightarrow f^{(n)}(0) = \frac{(-1)^{n-1}\, 1\times3\times5\times\ldots\times(2n-3)}{2^n}$$

$$\Rightarrow \sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\times3\times5\times\ldots\times(2n-3)}{2^n \times n!} x^n.$$

Take a_n as $\frac{(-1)^{n-1}\, 1\times3\times5\times\ldots\times(2n-3)}{2^n\times n!} x^n$, then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ is:

$$\lim_{n\to\infty}\left|\frac{1\times3\times5\times\ldots\times(2n-3)(2n-1)x^{n+1}}{2\times2^n\times(n!)(n+1)} \times \frac{2^n\times n!}{1\times3\times5\times\ldots\times(2n-3)x^n}\right| = \frac{|x|}{2}\lim_{n\to\infty}\left(\frac{2n-1}{n+1}\right) =$$

$$|x| < 1 \Rightarrow \underline{R=1}$$



As n increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $|x|<1$.

8.7:34.

$$\int e^{x^3}dx = \int \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)\,n!}$$

8.7:38.

$$\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty} \left[\frac{(-1)^n x^{2n+3}}{n!\,(2n+3)}\right]_0^{0.5=\frac{1}{2}} =$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+3)2^{2n+3}}$$

Since $c_2 = \frac{1}{1792} < 0.001$, take $\sum_{n=0}^{1} \frac{(-1)^n}{n!\,(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \simeq 0.0354$

8.8:1.

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{2^2 \times 2!} + \frac{1\times 3\times x^3}{2^3\times 3!} - \frac{1\times 3\times 5\times x^4}{2^4\times 4!} + \ldots$$

$$= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1\times 3\times 5\times\ldots\times(2n-3)\, x^n}{2^n \times n!}$$, $R=1$ (by Ratio Test)

4. $(1+x^2)^{\frac{1}{3}} = \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} x^{2n} = 1 + \frac{x^2}{3} + \frac{(\frac{1}{3})(-\frac{2}{3})}{2!} x^4 + \frac{(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})}{3!} x^6 + \dots$

$$= 1 + \frac{x^2}{3} + \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\, 2\cdot 5\cdot 8\cdot \dots \cdot (3n-4)\, x^{2n}}{3^n \times n!}, \quad R=1$$

9. a) $(1+(-x^2))^{-\frac{1}{2}} = 1 + (-\frac{1}{2})(-x^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-x^2)^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}(-x^2)^3 + \dots$

$$= 1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)}{2^n \times n!} x^{2n}$$

b) $\sin^{-1} x = \int \frac{1}{\sqrt{1-x^2}}\,dx = \int (1+(-x^2))^{-\frac{1}{2}}\,dx = C + x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)}{(2n+1)\,2^n \times n!} x^{2n+1}$

$$= x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-1)}{(2n+1)\,2^n \times n!} x^{2n+1} \quad \text{as } 0 = \sin^{-1}(0) = C$$

16. a) $4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \left[1+(-k^2\sin^2 x)\right]^{-\frac{1}{2}} dx =$

$$= 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \left[1 - \frac{1}{2}(-k^2\sin^2 x) + \frac{(\frac{1}{2})(\frac{3}{2})}{2!}(-k^2\sin^2 x)^2 - \frac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{3!}(-k^2\sin^2 x)^3 + \dots\right] dx =$$

$$= 4\sqrt{\frac{L}{g}} \int_0^{\frac{\pi}{2}} \left[1 + \left(\frac{1}{2}\right)k^2\sin^2 x + \left(\frac{1}{2}\times\frac{3}{4}\right)k^4\sin^4 x + \left(\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\right)k^6\sin^6 x + \dots\right] dx$$

(Split up the integral and use 5.6.36 results)

$$= 4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2} + \left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2 + \left(\frac{1\times 3}{2\times 4}\right)\left(\frac{1\times 3}{2\times 4}\times\frac{\pi}{2}\right)k^4 + \left(\frac{1\times 3\times 5}{2\times 4\times 6}\right)\left(\frac{1\times 3\times 5}{2\times 4\times 6}\times\frac{\pi}{2}\right)k^6 + \dots\right]$$

$$= 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2\times 3^2}{2^2\times 4^2}k^4 + \frac{1^2\times 3^2\times 5^2}{2^2\times 4^2\times 6^2}k^6 + \dots\right]$$

b) Since all of the terms are positive, the first inequality is true.

For the second:

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1^2}{2^2}k^2 + \frac{1^2\times 3^2}{2^2\times 4^2}k^4 + \frac{1^2\times 3^2\times 5^2}{2^2\cdot 4^2\cdot 6^2}k^6 + \dots\right] \leqslant 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{1}{4}k^4 + \dots\right.$$

$$2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}k^2 + \frac{1}{4}k^4 + \frac{1}{4}k^6 + \dots\right] = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{(k^2/4)}{1-k^2}\right] \geqslant T$$

c) Take $L=1$, $g=9.8$, $k=\sin(10^\circ/2) \approx 0.08716$, then the inequality from (b) becomes $2.01090 \leqslant T \leqslant 2.01093$, so $T \approx 2.0109$. Estimate $T \approx 2\pi\sqrt{\frac{L}{g}} \approx 2.0071$.

If $\theta_0 = 42^\circ$, then $k \approx 0.35837$ and the inequality from (b) becomes $2.07153 \leqslant T \leqslant 2.08103$, and $T \approx 2.0763$.

The discrepancy increases from about 0.2% to 3.4%.