Problem Set 8 (Tues/Thurs)

19) All derivatives of e^x are e^x, so
$|R_4(x)| \le \frac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$,
$R_n(0.1) \le \frac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$,
and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus we need four terms of the MacLaurin series ($n = 0, 1, 2, 3$). (in fact the sum is $1.1051\bar{6}$ and $e^{0.1} \approx 1.10517$.)

20) The Maclaurin series for $\ln(1+x)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x)^n}{n}$ for $|x| < 1$.
$\ln(1.4) = \ln(1+0.4) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(0.4)^n}{n}$. This is an alternating series. Using the Alternating Series Estimation Theorem,
$|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (nonzero) terms of the Maclaurin series.

22) $\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots$
Using the Alternating Series Estimation Theorem:
$|-\frac{1}{6!}x^6| < 0.005 \Rightarrow |x| < (3.6)^{1/6} \approx 1.238$. Graph: One should graph $y = \cos(x) + 0.005$ and $y = \cos(x) - 0.005$, and see where $y = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$ crosses outside (see picture p. 636 of solutions manual). Since cosine and our approximation function are even, need only to check $x > 0$. Range: $-1.238 < x < 1.238$.

Chapter 8 Exercises p 641: 40) Use the Maclaurin series for e^x, which we know:
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow xe^{2x} = x\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}$.
The radius of convergence for the Maclaurin series for e^x is $R = \infty$; it is the same for this series.

Plus:

Handout [redacted]

1) $f(x) = x^{1/3}$ at $x = 27$.
(a)

n	$f^{(n)}(x)$	$f^{(n)}(27)$
0	$x^{1/3}$	3
1	$\frac{1}{3}x^{-2/3}$	$1/3^3 = 1/27$
2	$\frac{1}{3}(-\frac{2}{3})x^{-5/3}$	$-2/3^7$
3	$\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})x^{-8/3}$	$10/3^{11}$
4	$\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})x^{-11/3}$	$-80/3^{15}$

$T_3(x) = 3 + \frac{1}{27}(x-27) + \frac{(-2/3^7)}{2!}(x-27)^2 + \frac{(10/3^{11})}{3!}(x-27)^3$
$T_3(x) = 3 + \frac{1}{27}(x-27) - \frac{1}{3^7}(x-27)^2 + \frac{5}{3^{12}}(x-27)^3$
(b)
$28^{1/3} \approx T_3(28) = 3 + \frac{1}{27}(1) - \frac{1}{3^7}(1)^2 + \frac{5}{3^{12}}(1)^3$
$= 3 + \frac{1}{27} - \frac{1}{3^7} + \frac{5}{3^{12}} \approx 3.03658920$.
(c) $|\text{Error}| \le |a_4| = |\frac{-80/3^{15}}{4!}(1)^4| \approx 2.32306 \times 10^{-7}$
(d)

For $27 \le x \le 28$, $|f^{(4)}(x)| \le |f^{(4)}(27)| = 80/3^{15}$. So $M = 80/3^{15}$ and by Taylor's Inequality, $|R_4(28)| \le \frac{M}{4!}(1)^4 \approx 2.32306 \times 10^{-7}$.

2) $\ln(1+u), -1 < u \le 1$. (a) As in problem 20 above, The Maclaurin series for $\ln(1+u)$ is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(u)^n}{n}$.
Among other methods, this can be derived from $\frac{1}{1-(-u)}$ by integration.
(b) Letting $u = x - 1$, a power series for $\ln(x)$ centered at $x = 1$ is
$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.
(c)
As in problem 3 above,

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$\frac{1}{x}$	1
2	$\frac{(-1)}{x^2}$	-1
3	$\frac{(-1)(-2)}{x^3}$	2
4	$\frac{(-1)(-2)(-3)}{x^4}$	-6

We can see that for $n \ge 1$, $f^{(n)}(1) = \frac{(-1)^{n+1}(n-1)!}{(1)^n}$. $f(1) = 0$, so the 0th term is 0. So the Taylor Series for $\ln(x)$ at $x = 1$ is
$ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$.
This agrees with the answer in part (b).

3) a) $\sin x \quad x \quad \frac{x}{3} +$

$\cos x \quad \frac{x}{2!} + \frac{x}{4!}$

N constant term

x-ter $\quad x \quad x$

No term

x term is $\quad 1 \cdot \frac{}{3!} \quad \frac{x}{2!} \cdot x \left(\frac{1}{} \right) x \quad \frac{2}{3} x$

No x-term

So $\sin x \cos x \quad x \quad \frac{2}{3} \quad +$

Check $\sin x \cos x = \frac{1}{2} \sin 2x = \frac{1}{2} \left(2x \quad \frac{(2x)^3}{3!} + \right) \quad x \quad \frac{2}{3} x + \quad \checkmark$

b) $\sqrt{1+x} \quad \sum_{n=1}^{\infty} \frac{(\frac{1}{2})(\frac{1}{2}-1)\ldots(\frac{1}{2}-n+1)}{n} x^n \quad \sqrt{1+x} \quad \sum_{n=1}^{\infty} \frac{(\frac{1}{2})(\frac{1}{2}-1)\ldots(\frac{1}{2}-n+1)}{n} x$

So multiplying by self gives (as first terms of $\sqrt{1+x}$ are $1 + \frac{x}{2} \quad \frac{x^2}{8}$

$1 + x + 0x^2 \quad 1+x$

The coefficient of x will be 0 as $1+x$ is equa to the entire taylor expansion.

Do: §8.9 # 23, 25 (Read the notes on Handout C for guidance)
As a start to your exam review, do the Concept Check: pg. 640 #3, 4, 10, 11a-d, and the true/false quiz # 1-7 and 11. Plus: Problem 3 from Handout C.

23) Let $s(t)$ be the position of the car as a function of time t. For convenience set $s(0) = 0$. The velocity is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$. So the Second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \frac{a(0)}{2}t^2 = 20t + t^2$. So $s(1) \approx T_2(1) = 20 + 1 = 21$ m.
$T_2(t)$ would not be accurate over a full minute because the car could not maintain an acceleration of $2m/s^2$. (Approximating for a minute later is analogous to moving farther away from the center of the Taylor polynomial, and we know our accuracy decreses. We would need to know something about how the acceleration changes in order to be accurate over a longer period of time, i.e. we need knowledge of higher order derivatives).

25) Using the note on handout C:

$$E = \frac{q}{D^2} - \frac{q}{(D+d)^2} = \frac{q}{D^2} - \frac{q}{D^2}\left[\frac{1}{\left(1+\frac{d}{D}\right)^2}\right] = \frac{q}{D^2}\left[1 - \frac{1}{\left(1+\frac{d}{D}\right)^2}\right] = \frac{q}{D^2}\left[1 - \left(1+\frac{d}{D}\right)^{-2}\right]$$

¿From this we can use the Binomial Series for $\left(1+\frac{d}{D}\right)^{-2}$, so:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{(2)(3)}{2!}\left(\frac{d}{D}\right)^2 - \frac{(2)(3)(4)}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right] = \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \quad\right] \approx \frac{q}{D^2}2\left(\frac{d}{D}\right) = 2qd\frac{1}{D^3}$$

when D is much larger than d, i.e. when P is far away from the dipole. This is because $\frac{d}{D} \gg (\frac{d}{D})^2 \gg (\frac{d}{D})^3$ when $\frac{d}{D}$ is very small.

Concept Check p. 640

3) (a) See (4) in Section 8.2.
A geometric series is a series of the form
$\sum_{n-1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$, Which is convergent if $|r| < 1$. It's sum is $\frac{a}{1-r}$. If $|r| \geq 1$, the geometric series is divergent.
(b) See (1) in Section 8.3.
The p-series has the form
$\sum_{n-1}^{\infty} \frac{1}{n^p}$, which converges if $p > 1$ and diverges if $p \leq 1$.

4) $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} s_n = 3$.

10) (a) $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$
(b) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$
(c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$ ($a = 0$ in part (b))
(d) See Theorem 8.7.8.
If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, then to show $f(x)$ is equal to the sum of its Taylor series for $|x-a| < R$, one must show that $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$.
(e) See Taylor's Inequality (8.7.9).
If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ for $|x-a| \leq d$.

11) See table page 618 of book.

True-False Quiz

1) False. (More accurately, could be true, but not necessarily, for example, the harmonic series.)

2) True. Theorem 8.5.3: The series are based on a power series $\sum c_n x^n$ centered at $x = 0$ with the radius of convergence R at least 6. Since -2 is in the interval of convergence, the second series converges. Or use the comparison test, showing that $\sum c_n(-2)^n$ converges absolutely.

3) False. (could be true but not necessarily) Example: take $c_n = (-1)^n/(n6^n)$.

4) True. Theorem 8.5.3. Similar to question 2 above, except 10 is outside the interval of convergence, so the series diverges.

5) False. Using the Ratio Test, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{n^3}{(n+1)^3} = 1$, so nothing can be conlcuded.

6) True. Using the Ratio Test, $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{n!}{(n+1)!}| = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$.

7) False. (could be true) We cannot conclude anything about $\sum a_n$ since it is smaller than a divergent series.

11) True. By Theorem 8.7.5 the coefficient of x^3 is $\frac{f'''(0)}{3!} = \frac{1}{3} \Rightarrow f'''(0) = 2$.
Or: Use Theorem 8.6.2 to differentiate f three times.

Plus:

4) Handout D

(a) We know Charlie's statement is false because we know $1 + (-1) + (-1)^2 + (-1)^3 + (-1)^4 + ...$ diverges. We cannot plug into the formula $\frac{a}{1-r}$ because the formula only applies for $|r| < 1$, which is not true in this case. To show the series diverges, we look at the nth partial sum, $S_n = \frac{a(1-r^n)}{1-r}$, and take the limit as n approaches infinity: $\lim_{n\to\infty} S_n = \frac{1-(-1)^n}{2}$ does not exist, so the series diverges.

(b) Charlie's closed form is incorrect because if $r = 7^2$, n should equal 10, not 20. The correct sum is given by $\frac{a(1-r^{n+1})}{1-r} = \frac{3(1-49^{11})}{1-49}$.
Charlie's statement about convergence is not accurate because Amanda's expression is not an infinite series (or it is an infinite series but with the terms all zero after the 11th term). The expression has a finite sum. Charlie's statement would be correct if the expression was an infinite geometric series with $r = 49$.

(c) Charlie's advice is not solid. It is true that if a_k does not approach zero as k approaches infinity, the series diverges, but if it does approach zero, we cannot make a conclusion about the convergence of the series. For example, p-series converge for $p > 1$ but diverge for $0 < p \leq 1$, even though in both cases $\lim_{k\to\infty} a_k = 0$.