

# MATH 1 B (FALL '95)

## SOLUTIONS TO THE FINAL EXAM

① a)  $\int \sec^2 \theta \, d\theta = \boxed{\tan \theta + C}$  (since  $\frac{d}{d\theta} \tan \theta = \sec^2 \theta$ )

b)  $\int z e^{3z^2+1} \, dz = \frac{1}{6} \int e^w \, dw = \frac{1}{6} e^w + C$   
 $w = 3z^2 + 1$   
 $dw = 6z \, dz$  so  
 $z \, dz = \frac{dw}{6}$   
 $= \boxed{\frac{1}{6} e^{3z^2+1} + C}$

c)  $\int y \sec^2(y) \, dy = y \tan y - \int \frac{\sin y}{\cos^2 y} \, dy$   
 $= y \tan y - \ln |\cos y| + C$   
 (set  $z = \cos y$   
 $dz = -\sin y \, dy$ )  
 $u = y \quad v = \sec^2 y$   
 $u' = 1 \quad v' = 2 \sec y \tan y$

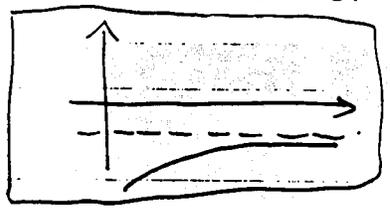
d)  $\int \frac{dt}{1 + \sqrt{t}} = 2 \int \left( \frac{u-1}{u} \right) du = 2 \int \left( 1 - \frac{1}{u} \right) du$   
 $= 2u - 2 \ln |u| + C$   
 $= \boxed{2(1 + \sqrt{t}) - 2 \ln(1 + \sqrt{t}) + C}$   
 $u = 1 + \sqrt{t}$  so  
 $(u-1)^2 = t$  so  
 $2(u-1)du = dt$

e)  $\int \frac{dx}{(b+ax)^2} = \frac{1}{a} \frac{du}{u^2} = \frac{-\frac{1}{a} u^{-1} + C}{a}$   
 $= \boxed{-\frac{1}{a} (b+ax)^{-1} + C}$   
 $u = b+ax$   
 $du = a \, dx$   
 $dx = \frac{du}{a}$

- ② THE POLLUTANTS ARE LEAVING THE LAKE AND NO NEW POLLUTANTS ARE ENTERING SO THE NET RATE IS THE RATE OUT! THE TOTAL VOLUME OUT IS 2 gals/min OF WHICH ONLY  $2 \cdot \frac{Q}{V}$  IS THE POLLUTANT (VOLUME  $\times$  CONCENTRATION), SO

$$\boxed{\frac{dQ}{dt} = -2 \frac{Q}{V} \text{ OR CHOOSE C)}$$

③ SINCE  $f(y) \geq 0$  FOR ALL  $y$ ,  $\frac{dy}{dx} \geq 0$  FOR ALL  $y$   
 SO  $y(x)$  IS NONDECREASING FOR ALL  $y$  HENCE CHOOSE



④  $\cos t = 1 - \frac{t^2}{2!} + \dots$  SO  $\cos(3x^2) = 1 - \frac{(3x^2)^2}{2!} + \dots$   
 $= 1 - \frac{9}{2}x^4 + \dots$

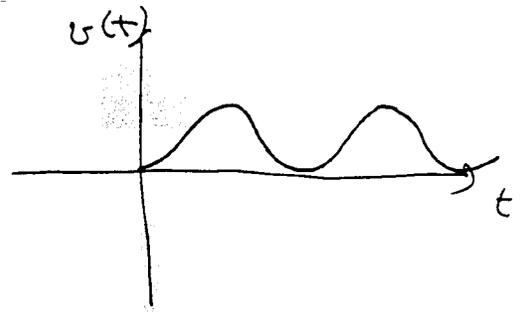
⑤ IN GENERAL,  $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$   
 SO IF  $a = 2$ , WE HAVE

$f(x) \approx P_3(x) = 4 + 3(x-2) - \frac{5}{2}(x-2)^2 + \frac{12}{6}(x-2)^3$   
 $= 4 + 3(x-2) - \frac{5}{2}(x-2)^2 + 2(x-2)^3$

⑥ FROM THE PICTURE, WE SEE THAT  $f(a) > 0$ ,  
 $g'(a) > 0$ ,  $g''(a) < 0$  SO  
 a) POSITIVE      b) POSITIVE      c) NEGATIVE

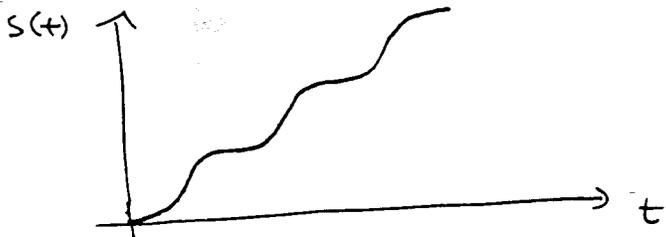
(AS  $g(x) \approx g(a) + g'(a)(x-a) + \frac{g''(a)}{2!}(x-a)^2$  SO  
 $C_0 = g(a)$ ,  $C_1 = g'(a)$ ;  $C_2 = \frac{g''(a)}{2}$ )

⑦ a)  $v(0) = 0$  AND  
 $v(t) \geq 0$  SINCE FOR NO  $t$   
 IS THE AREA BETWEEN 0  
 AND  $t$  UNDER  $a(t)$   
 NEGATIVE



7 (CONTINUED)

IF  $s(t)$  IS THE POSITION OF THE PARTICLE, WE ARE GIVEN THAT  $s(0) = 0$  AND WE KNOW THAT  $s(t)$  IS NONDECREASING SINCE  $v(t) \geq 0$  FOR ALL  $t$  SO



8 TOTAL SPENDING (IN MILLIONS OF DOLLARS) IS

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$  THIS IS A CONVERGENT

GEOMETRIC SERIES WITH  $a = 1$ ,  $x = \frac{1}{2}$  SO ITS SUM

$S = \frac{a}{1-x} = \frac{1}{1-\frac{1}{2}} = 2$  MILLION DOLLARS

9 a)

COST = (COST PER KW HOUR) \* (NUMBER OF KW) \* (HOURS) <sup>IF</sup> BOTH OF THE LATTER QUANTITIES ARE CONSTANT (WHICH THEY ARE NOT!)

SO WE PICK SOME TIME  $t_i$  AND ASSUME THAT FOR TIMES BETWEEN  $t_i$  AND  $t_i + \Delta t$  THE ABOVE QUANTITIES ARE APPROXIMATELY CONSTANT ON THE SMALL TIME INTERVAL. SO LET  $C_i$  BE THE  $i$ th CONTRIBUTION TO THE COST THEN

$C_i \approx E(t_i) C(t_i) \Delta t = [5 + \sin(\frac{\pi t_i}{12})] [0.1 + 0.01 \sin(\frac{\pi t_i}{24})] \Delta t$

THEN COST  $\approx \sum_{i=1}^n C_i = \sum_{i=1}^n E(t_i) C(t_i) \Delta t = \sum_{i=1}^n [5 + \sin(\frac{\pi t_i}{12})] [0.1 + 0.01 \sin(\frac{\pi t_i}{24})] \Delta t$

1) LETTING  $n \rightarrow \infty$  WE GET

COST =  $\int_0^{24} [5 + \sin(\frac{\pi t}{12})] [0.1 + 0.01 \sin(\frac{\pi t}{24})] dt$

(10)

a)

SEPARATING VARIABLES WE GET

$$\frac{dy}{y} = \cos x \, dx \quad \text{or} \quad \int \frac{dy}{y} = \int \cos x \, dx$$

$$\text{so } \ln|y| = \sin x + C \quad \text{or } |y| = e^C \cdot e^{\sin x}$$

$$\text{or } y = \pm e^C e^{\sin x} \quad \text{or } y = A e^{\sin x}$$

$$\text{SINCE } y(0) = A = 4 \text{ WE GET } \boxed{y = 4e^{\sin x}}$$

b) SINCE  $e^t$  IS AN INCREASING FUNCTION IT WILL REACH ITS MINIMUM AT THE MINIMAL VALUE OF ITS INPUT AND IT WILL REACH ITS MAXIMUM AT THE MAXIMAL VALUE OF ITS INPUT SO THE MINIMUM IS REACHED WHEN  $\sin x = -1$  AND

$$\boxed{\text{MIN} = 4e^{-1} = \frac{4}{e}}$$

AND MAXIMUM IS

REACHED WHEN  $\sin x =$ 

AND

$$\boxed{\text{MAX} = 4e}$$

(11)

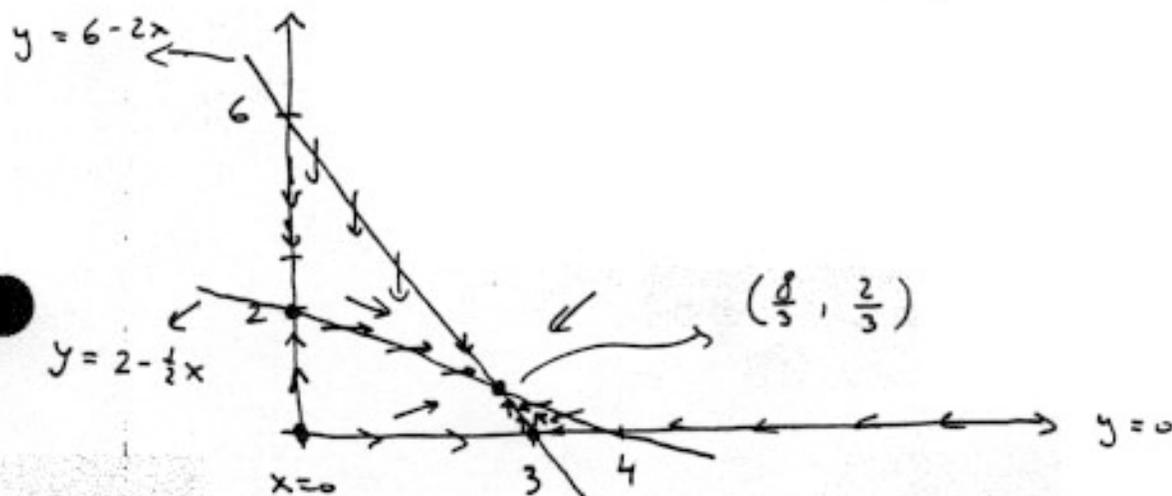
a)  $\frac{dx}{dt} = x(6 - 2x - y)$  so  $\frac{dx}{dt} = 0$  IF

$$\boxed{x = 0 \quad \text{OR} \quad y = 6 - 2x}$$

$$\frac{dy}{dt} = y(4 - 2y - x)$$
 so  $\frac{dy}{dt} = 0$  IF

$$\boxed{y = 0 \quad \text{OR} \quad y = 2 - \frac{1}{2}x}$$

SO



(5)

b) THE EQUILIBRIA ARE THE POINTS WHERE BOTH  $\frac{dx}{dt}$  AND  $\frac{dy}{dt}$  ARE ZERO SO

$$(0, 0), (0, 2), (3, 0), \left(\frac{8}{3}, \frac{2}{3}\right)$$

c) i) THEY COMPETE

d) IN THE LONG RUN THE TWO POPULATIONS TEND TO  $\left(\frac{8}{3}, \frac{2}{3}\right)$ , REGARDLESS OF INITIAL CONDITIONS.

(12)

a) F C = 6 THE CHARACTERISTIC EQUATION IS

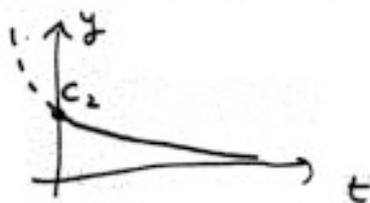
$$2\lambda^2 + 6\lambda + 4 = 0 \quad \text{OR} \quad 2(\lambda + 2)(\lambda + 1) = 0$$

SO  $\lambda = -2$  OR  $\lambda = -1$ ; HENCE

$$y(t) = C_1 e^{-2t} + C_2 e^{-t}$$

A POSSIBLE SOLUTION (IF  $C = 0$ ) IS  $C_2 e^{-t}$

OR



b) IF C = 5 THEN THE CHARACTERISTIC EQUATION IS

$$2\lambda^2 + 5\lambda + 4 = 0 \quad \text{OR}$$

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{25 - 32}}{4} = \frac{-5 \pm \sqrt{-7}}{4} = -\frac{5}{4} \pm \frac{\sqrt{7}}{4} i$$

$$\text{SO } y(t) = C_1 e^{-\frac{5}{4}t} \cos\left(\frac{\sqrt{7}}{4}t\right) + C_2 e^{-\frac{5}{4}t} \sin\left(\frac{\sqrt{7}}{4}t\right)$$

(6)

(12) CONTINUED

c) FOR  $0 \leq c < \sqrt{32}$  SINCE WE WANT  $\sqrt{c^2 - 32}$  TO BE COMPLEX i.e.

$c^2 - 32 < 0$ , THIS IS TRUE FOR

$-\sqrt{32} < c < \sqrt{32}$ , BUT SINCE WE ARE GIVEN THAT  $c \geq 0$ , WE CONCLUDE

$$0 \leq c < \sqrt{32}$$

d) SUPPOSE  $c$  IS BETWEEN 0 AND  $\sqrt{32}$ . THEN THE CHARACTERISTIC EQUATION

$2N^2 + cN + 4 = 0$  HAS SOLUTIONS

$$\frac{-c \pm \sqrt{c^2 - 32}}{4} \quad \text{OR} \quad \frac{-c \pm \sqrt{32 - c^2}i}{4}$$

SO THE SOLUTIONS TO THE DIFF. EQN ARE OF THE FORM

$$y(t) = C_1 e^{-\frac{c}{4}t} \cos\left(\frac{\sqrt{32 - c^2}}{4}t\right) + C_2 e^{-\frac{c}{4}t} \sin\left(\frac{\sqrt{32 - c^2}}{4}t\right)$$

i) THE PERIOD =  $\frac{2\pi}{\frac{\sqrt{32 - c^2}}{4}} = \frac{8\pi}{\sqrt{32 - c^2}}$

ii) PERIOD IS MINIMIZED BY THE LARGEST POSSIBLE DENOMINATOR i.e. BY  $c = 0$  SO  $\frac{8\pi}{\sqrt{32}}$  IS THE MINIMAL PERIOD

iii) WANT  $\frac{8\pi}{\sqrt{32 - c^2}} = 0$  SO  $\frac{64\pi^2}{100} = 32 - c^2$  SO  $c \approx 5.07$